THE 2×3 ARROW IMPOSSIBILITY THEOREM

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ABSTRACT. A proof of Arrow's Impossibility Theorem is presented for a case of two individuals and three options.

Individuals a and b have preferences over three options x_1, x_2 and x_3, and preferences are irreflexive simple orders (transitive, and such that for any i, j one and one only of $x_i \succ x_j$ or $x_j \succ x_i$ holds). We let Π be the six-element set of possible preferences.

Definitions. A *Social Welfare Function* (SWF) is an $f \colon \Pi^2 \to \Pi$. A SWF satisfies *Independence of Irrelevant Alternatives* (IIA) if the relative ranking of any two options in $f(\succ_a, \succ_b)$ depends only on the ranking of the two options in $\succ_a$ and $\succ_b$. A SWF satisfies *Unanimity* (U) if $f(\succ_a, \succ_b)$ ranks x_i higher than x_j whenever both $\succ_a$ and $\succ_b$ do. A SWF is *Dictatorial* if $f(\succ_a, \succ_b) = \succ_a \ \forall \succ_b$ or $f(\succ_a, \succ_b) = \succ_b \ \forall \succ_a$.

Theorem. *A SWF which satisfies U and IIA is dictatorial.*

Proof. There are pairs x_i, x_j for which a and b's preferences disagree; we show that if for one pair (i, j) $x_i \succ_a x_j$ implies that $f(\succ_a, \succ_b) \equiv \succ_s$ has $x_i \succ_s x_j$ even if $x_j \succ_b x_i$, then the same holds for all the six (i, j)'s.

So suppose that $x_1 \succ_a x_2$ implies $x_1 \succ_s x_2$. Then, firstly $x_i \succ_a x_j$ implies $x_i \succ_s x_j$ whenever x_1 is on the left or x_2 on the right. For, assume $x_1 \succ_a x_3$; to show that even if $x_3 \succ_b x_1$ one has $x_1 \succ_s x_3$: since by IIA the relative ranking of x_1, x_3 in $\succ_s$ is independent of the position of x_2, it suffices to observe that $f(x_1 \succ x_2 \succ x_3\, ,\, x_2 \succ x_3 \succ x_1)$ has $x_1 \succ_s x_2$ by hypothesis and $x_2 \succ_s x_3$ by unanimity, whence $x_1 \succ_s x_3$ by transitivity. The same argument holds if $x_3 \succ_a x_2$, by taking $f(x_3 \succ x_1 \succ x_2\, ,\, x_2 \succ x_3 \succ x_1)$ which must have $x_3 \succ_s x_2$.

Building on what is proven, for the remaining three cases, $x_2 \succ_a x_3$, $x_2 \succ_a x_1$ and $x_3 \succ_a x_1$ we can take in turn $f(x_2 \succ x_1 \succ x_3\, ,\, x_3 \succ x_2 \succ x_1)$, $f(x_2 \succ x_3 \succ x_1\, ,\, x_3 \succ x_1 \succ x_2)$ and $f(x_3 \succ x_2 \succ x_1\, ,\, x_2 \succ x_1 \succ x_3)$. $\square$

REFERENCE

[1] Arrow, K. (1951, 1963). *Social Choice and Individual Values*, 1st. and 2nd. ed., New York: Wiley.