

of the test scores. Thus,

$$X = \max\{X_1, X_2, X_3\},$$

where X_1, X_2, X_3 are the three test scores and X is the final score. Assume that your score in each test takes one of the values from 1 to 10 with equal probability $1/10$, independently of the scores in other tests. What is the PMF p_X of the final score?

We calculate the PMF indirectly. We first compute the CDF F_X and then obtain the PMF as

$$p_X(k) = F_X(k) - F_X(k-1), \quad k = 1, \dots, 10.$$

We have

$$\begin{aligned} F_X(k) &= \mathbf{P}(X \leq k) \\ &= \mathbf{P}(X_1 \leq k, X_2 \leq k, X_3 \leq k) \\ &= \mathbf{P}(X_1 \leq k) \mathbf{P}(X_2 \leq k) \mathbf{P}(X_3 \leq k) \\ &= \left(\frac{k}{10}\right)^3. \end{aligned}$$

where the third equality follows from the independence of the events $\{X_1 \leq k\}$, $\{X_2 \leq k\}$, $\{X_3 \leq k\}$. Thus, the PMF is given by

$$p_X(k) = \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3, \quad k = 1, \dots, 10.$$

The preceding line of argument can be generalized to any number of random variables $X_1, \dots, X_n$. In particular, if the events $\{X_1 \leq x\}, \dots, \{X_n \leq x\}$ are independent for every x , then the CDF of $X = \max\{X_1, \dots, X_n\}$ is

$$F_X(x) = F_{X_1}(x) \cdots F_{X_n}(x).$$

From this formula, we can obtain $p_X(x)$ by differencing (if X is discrete), or $f_X(x)$ by differentiation (if X is continuous).

The Geometric and Exponential CDFs

Because the CDF is defined for any type of random variable, it provides a convenient means for exploring the relations between continuous and discrete random variables. A particularly interesting case in point is the relation between geometric and exponential random variables.

Let X be a geometric random variable with parameter p ; that is, X is the number of trials until the first success in a sequence of independent Bernoulli trials, where the probability of success at each trial is p . Thus, for $k = 1, 2, \dots$, we have $\mathbf{P}(X = k) = p(1-p)^{k-1}$ and the CDF is given by

$$F_{\text{geo}}(n) = \sum_{k=1}^n p(1-p)^{k-1} = p \frac{1 - (1-p)^n}{1 - (1-p)} = 1 - (1-p)^n, \quad \text{for } n = 1, 2, \dots$$

Suppose now that X is an exponential random variable with parameter $\lambda > 0$. Its CDF is given by

$$F_{\text{exp}}(x) = \mathbf{P}(X \leq x) = 0, \quad \text{for } x \leq 0,$$

and

$$F_{\text{exp}}(x) = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^x = 1 - e^{-\lambda x}, \quad \text{for } x > 0.$$

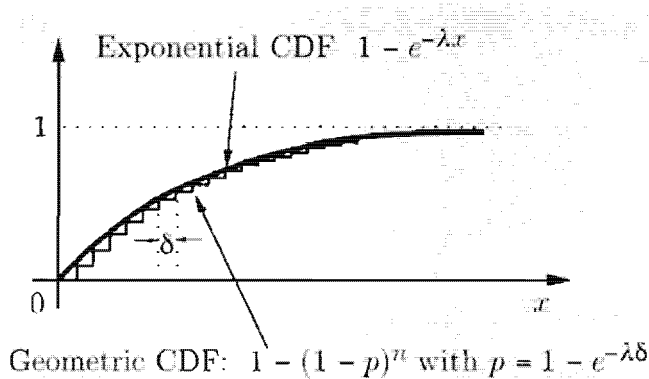


Figure 3.8: Relation of the geometric and the exponential CDFs. We have

$$F_{\text{exp}}(n\delta) = F_{\text{geo}}(n), \quad n = 1, 2, \dots,$$

where δ is chosen so that $e^{-\lambda\delta} = 1 - p$. As δ approaches 0, the exponential random variable can be interpreted as a “limit” of the geometric.

To compare the two CDFs above, let us define $\delta = -\ln(1 - p)/\lambda$, so that

$$e^{-\lambda\delta} = 1 - p.$$

Then, we see that the values of the exponential and the geometric CDFs are equal whenever $x = n\delta$, with $n = 1, 2, \dots$, i.e.,

$$F_{\text{exp}}(n\delta) = F_{\text{geo}}(n), \quad n = 1, 2, \dots$$

and are close to each other for other values of x (see Fig. 3.8). Suppose now that we toss very quickly (every δ seconds, where $\delta \ll 1$) a biased coin with a very small probability of heads (equal to $p = 1 - e^{-\lambda\delta}$). Then, the first time to obtain a head (a geometric random variable with parameter p) is a close approximation to an exponential random variable with parameter λ , in the sense that the corresponding CDFs are very close to each other, as shown in Fig. 3.8. This relation between the geometric and the exponential random variables will play an important role when we study the Bernoulli and Poisson processes in Chapter 6.