

# Failing to Provide Public Goods: Why the Afghan Army did not Fight<sup>☆</sup>

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## Abstract

The theory of public goods is mainly about the difficulty in paying for them. Here we ask why when funding is available public goods may never-the-less not be provided as in the case of the Afghan Army. We explore this issue using a simple model of a public good that can be provided through collective action and peer pressure by modeling the self organization of a group (the Afghan Army) as a mechanism design problem. We consider two kinds of transfer subsidies from an external entity such as the US government. One is a Pigouvian subsidy that simply pays the salaries rewarding individuals who provide effort. The second is an output/resource multiplier (for example, the provision of military equipment, tactical skills, training, and so forth) that amplifies the effort provided through collective action. We show that the introduction of a Pigouvian subsidy can result in less effort being provided than in the absence of a subsidy. By contrast, an output/resource multiplier subsidy, which is useful only if collective action is taken, necessarily increases output via an increase in effort. Our conclusion is that the fact that the USA provided the wrong kind of subsidy may have been among the reasons why the Afghan army did not fight. *JEL Classification Numbers:* A1, D7, D9

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We paid their salaries...What we could not provide was the will to fight..." Joseph Biden

## 1. Introduction

The speed with which the Afghan army collapsed and the Taliban took over Afghanistan came as a surprise to many but not to economists or individuals versed in game theory. By backward induction, if you plan to surrender anyway, then sooner is generally better than later unless the group is indifferent between present versus future payoffs. This analysis, however, begs the deeper question of why a well equipped army that outnumbered their opponents by three or four to one in manpower and with decades of training planned to surrender to an apparently much inferior opponent.

The quote from Joseph Biden's speech following the fall of Kabul is revealing: what Biden and many others fail to understand is that there is a causal connection between paying the salaries of the Afghan army and the fact that they lacked the will to fight. Our goal in this paper is to explain why that is so and why it need not have been so.

Insofar as nation building is measured by the national defense, the place to start is to understand that national defense is a public good. Many Afghans would prefer not to be ruled by the Taliban, but most would prefer that someone else do the fighting. This problem of free-riding is endemic to public goods problems and economists and other social scientists have analyzed these problems for over a century. We recognize three ways of overcoming the free-rider problem. The most familiar one involves collective action through formal systems, usually Pigouvian taxes or subsidies which are widely recommended by economists to achieve, for example, reductions in carbon emissions to combat global warming. A less formal means of providing public goods is through voluntary provision: people contribute to a public good either because the personal benefit of the public good is sufficiently great to outweigh the cost of contributing or because they are altruistic and desirous of helping their fellows (as in funding of public radio and television in the US with NPR and PBS). There is little evidence, however, that voluntary public goods provision can provide public goods on a large scale - for an entire country, for example as in the case of an army.

We wish to focus on a third means of providing public goods: incentives may be provided informally or "socially" through means such as peer pressure, resulting in various forms of ostracism of those who fail to contribute. Although economists have not studied this to the extent that they have studied taxes and subsidies we know, particularly from the work of Coase (1960), Ostrom (1990) and Townsend (1994) that these methods work in practice. Indeed: the effectiveness of large scale lobbying organizations such as farmers show that peer pressure can be effective even on a very large scale. After all, all farmers want the benefits of farm subsidies, but prefer that other farmers bear the cost of lobbying.

Of course while groups and societies can collectively self-organize social norms which induce provision of public goods, they may also choose simply to follow the "law of the jungle" and allow members to go their own way and free ride as they wish. The right choice depends on how valuable the public good is and how costly it is to organize and enforce collective decisions. The intervention

of outside agencies - be they NGOs or the USA - changes the trade-offs for collective decision making. Simply put: if the USA pays the salaries of the Afghan army then there is little benefit from the Afghans collectively organizing to encourage people to join the army and fight for their country. In practice, if the salary is sufficiently high relative to the outside option, these might join but will not fight when it's time to deliver. Indeed General Wesley Clark, former NATO supreme allied commander, gives the following description of the motivation of Afghan soldiers: "People signed up with the Afghan military to make money...but they did not sign up to fight to the death, for the most part."<sup>4</sup> Contrast this to J.R.R. Tolkien's description of Britain at the start of World War I: "In those days chaps joined up, or were scorned publicly." We think we may take it that such peer pressure to defend the country did not exist in Afghanistan.

Here is the key point: The displacement of self-organization by subsidy can result in less provision of the public good than in the absence of the subsidy. In other words: subsidizing a public good - the Pigouvian approach - can reduce the provision of that good if it displaces self-organization. The reason is that self-organization is costly and so the benefit of not organizing can exceed the cost of having less of the public good.

In this paper we examine a simple model of a public good that can be provided through collective action and peer pressure and examine the effect of subsidies. Our model follows Townsend (1994), Levine and Modica (2016) and Dutta, Levine and Modica (2021) by modeling the self organization of a group as a mechanism design problem. The group can establish an output quota, it has a noisy monitoring technology for observing whether the quota is followed, and it can punish group members based on these signals. A key feature of the model is that if monitoring and punishment is to be used it has an associated fixed cost: this includes both physical cost of monitoring and the costs of negotiating and finding an agreement as to what the mechanism will be.

In this setting we consider two kinds of subsidies. One is a Pigouvian subsidy that simply "pays the salaries" rewarding individuals who provide effort. We show that this can result in less effort being provided than in the absence of a subsidy. The second is an output multiplier: the provision of training, equipment, and so forth, that amplifies the effort provided through collective action. Because this is useful only if collective action is taken, unlike a Pigouvian subsidy it necessarily increases output.

Our earlier work Dutta, Levine and Modica (2021) showed more broadly how Pigouvian subsidies can have the perverse effect of undermining existing collective action. We pointed there to the case of NGOs. Bano (2012) did extensive field research in Pakistan. She documented how public goods, particularly welfare, were provided through voluntary efforts with socially provided incentives for contribution. Donor organizations - mostly NGOs - subsequently attempted to increase public good provision through subsidies in the form of salaries to those contributing to the public good. In a series of case studies she showed how this led to the unraveling of the provision of social

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<sup>4</sup>Quoted by CNN April 19, 2021, in the *Afghanistan-Taliban News* blog <https://edition.cnn.com/world/live-news/afghanistan-taliban-us-news-08-18-21>

incentives and ultimately to decreased provision of the public good. In one of several cases she indicates that “[t]he Maternity and Child Welfare Association... almost collapsed with the influx of such aid.”

The evidence now shows that similar considerations can be applied to the Afghan army. We cannot know how strong the social pressure to self-organize resistance to the Taliban would have been without subsidies; but the fact is that by paying the salaries of soldiers the incentive for collective action to encourage volunteers to join the Army for the common good was reduced so much that provision of the public good - measured not by the number of soldiers, but by the number of soldiers willing to fight - was minimal. Hence the Taliban, an army recruited through social incentives, predominates and once again rules Afghanistan.

The collapse of Afghanistan is often compared to the collapse of South Vietnam. In this context it is worth pointing out that the USA did not pay soldiers salaries in South Vietnam but only provided subsidies in the form of training and equipment. What is less well known is that as a result the South Vietnamese army (ARVN) did fight.<sup>5</sup> The USA withdrew from Vietnam in 1973. In the next year the ARVN largely drove the Vietcong, the North Vietnam irregular army somewhat akin to the Taliban, out of the South. In 1975 the North invaded with a large regular army of similar strength to the ARVN including a great many artillery pieces. The fighting lasted about four months and the casualties on both sides combined were about 45,000 killed and 80,000 wounded. This is greatly different from Afghanistan where a large superior well equipped military refused to fight and was defeated in weeks by a small lightly armed group of irregular fighters.

The bottom line is not entirely negative either for nation building or for NGOs. It is not that help cannot be provided, but care must be taken that the help provided does not undermine the provision of effort through collective action and social norms. Hence providing military training and equipment will generally result in greater defense, just as providing computers and training to charitable organizations can do the same.

## 2. The Model

Identical group members  $i \in [0, 1]$  engage in production choosing a real valued level of output  $X \geq x^i \geq 0$  such as fighting effort in the case of Afghanistan. The utility of a member  $i$  depends upon a vector valued state of the world  $\omega \geq 0$ , their own output, and the average output of the group  $x = \int x^i di$  according to  $u(\omega, x, x^i)$  where  $u(\omega, x, x^i)$  is specified below.

The output of the group  $x$  is a public good - in our case national defense. Because all members benefit from the public good the group collectively faces a mechanism design problem, and we assume that incentives can be given to group members in the form of individual punishments based on monitoring: the group can set a production quota  $y$  and receives signals of whether or not individual output adheres to the quota. Based on these signals it can impose punishments. Specifically, monitoring generates a noisy signal  $z^i \in \{0, 1\}$  where 0 means “good, likely respected

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<sup>5</sup>The historical facts are not controversial and are discussed in many histories, see, for example, Willbanks (2004).

the quota” and 1 means “bad, likely produced less than the quota.” The probability of the bad signal is  $\pi > 0$  if  $x^i \geq y$  and  $\pi_B > \pi$  if  $x^i < y$ . When the signal is bad the group imposes an endogenous utility penalty of  $P$ . This may be in the form of social disapproval or even in the form of monetary penalties.

The social cost of the punishment  $P$  is  $\psi P$  where  $\psi > 0$  could be greater or less than one. For example, if the punishment is that group members are prohibited from drinking beer with the culprit that might be costly to the culprit’s friends as well as the culprit. In this case  $\psi > 1$ . Or it might be that the punishment is a monetary fine most of which is shared among the group members. In that case there would be very little social loss so we would expect  $\psi < 1$ .

In addition to the social cost of punishment there is a fixed cost  $F \geq 0$  of choosing  $P > 0$ . There are two reasons we expect  $F$  to be positive. First, there will generally be costs of operating the monitoring system - for example, sending spies to observe output. Second, it is costly to gather together group members to negotiate an agreement and form a consensus on what the mechanism will be.

The tools available for mechanism design consist of a quota  $y$  and a punishment level for a bad signal  $P$ . The overall utility of a member  $i$  is  $u(\omega, x, x^i) - \pi P$  if  $x^i \geq y$  and  $u(\omega, x, x^i) - \pi_B P$  if  $x^i < y$ . These utilities define a game for the group members. If the mechanism designer chooses  $(y, P)$  we denote by  $\mathcal{X}(y, P)$  the set of  $x$  such that  $x^i = x$  is a symmetric pure strategy Nash equilibrium of this game. We refer to a triple  $(x, y, P)$  with  $x \in \mathcal{X}(y, P)$  as an *incentive compatible social norm*.<sup>6</sup> If an incentive compatible social norm issues no punishments ( $P = 0$ ) we call it *non-cooperative*. The mechanism designer is benevolent and welfare from an incentive compatible social norm  $(x, y, P)$  is given by

$$W(x, y, P) \equiv u(\omega, x, x) - \pi\psi P - F \cdot \mathbf{1}\{P > 0\}.$$

We now specify the utility function and interventions. Each individual has a private cost of output  $(c/2)(x^i)^2$  and the benefit of the public good is  $x - (1 - c)(1/2)x^2$ . These units are chosen so that the first best  $x^f$  maximizing  $-(c/2)x^2 + x - (1 - c)(1/2)x^2$  is normalized so that  $x^f = 1$ . We take the effort limit  $X$  to coincide with the satiation level for the public good gross benefit  $x - (1 - c)(1/2)x^2$ ; so  $X = 1/(1 - c)$ , and we assume that  $0 < c < 1$ .

A novel aspect of the present paper - which is what drives its main result - is the exact specification of the state variable. The state  $\omega$  has two components: a Pigouvian subsidy  $\omega_s$  which may be thought of as contributing to the salary of group members who provide effort, and an output multiplier  $\omega_m$  which may be thought of as equipment and training that increases the effectiveness of effort provided by group members. Overall individual utility is then given by  $u(\omega, x, x^i) = -(c/2)(x^i)^2 + \omega_s x^i + (1 + \omega_m)x - (1 - c)(1/2)((1 + \omega_m)x)^2$ . We are going to show that the two types of subsidies - in our case provided mainly by the USA - have quite different consequences in terms of effort provision in the organization.

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<sup>6</sup>In the language of contract theory it is an enforcement contract with costly state verification.

We define the *monitoring difficulty* as  $\theta = \psi\pi/(\pi_B - \pi)$ .

### 3. Subsidies Are Bad, Training is Good

We are interested in reversals, that is conditions under which introducing a subsidy reduces the effort level  $\hat{x}(\omega)$  that solves the mechanism design problem. Recalling that the group solves a mechanism design problem, we denote the optimal choice of  $x$  conditional on  $P > 0$  by  $x^M(\omega)$  and the output of the non-cooperative social norm (the Nash equilibrium) by  $x^N(\omega)$ . The solution to the design problem  $\hat{x}$  may be either  $x^M$  or  $x^N$ . Formally there is a *reversal* if  $(1 + \omega_m)\hat{x}(\omega) < \hat{x}(0)$ . By *no reversal* we mean the opposite inequality holds (strictly). We show in the Appendix that  $x^N(\omega)$  and  $x^M(\omega)$  are strictly increasing; and that for the relevant range of  $\omega$  it is  $x^M(0) > (1 + \omega_m)x^N(\omega)$ . It follows that the only way in which a reversal can occur is if the group prefers to use the mechanism  $M$  with punishment at  $\omega = 0$  but reverts to noncooperation at  $\omega$ . That is, the subsidy, salaries paid by the US for example, substitute for the costly use of peer pressure. Our main result states the conditions under which this may or may not occur:

**Theorem 1.** *For each  $\omega_s$  in a range  $0 < \omega_s < \bar{\omega}_s$  and for all sufficiently small  $F > 0$  and  $\omega_m \geq 0$  there is a reversal. On the contrary, for each  $\omega_m$  in a range  $0 < \omega_m < \bar{\omega}_m$  and for all  $F \geq 0$  and sufficiently small  $\omega_s \geq 0$  there is no reversal.*

Subsidies, in other words, are bad in the sense that they can reduce output, while training can only increase output. The result is proved in the Appendix.

### 4. Discussion

How does the model explain the collapse of the Afghan army? According to our theory the payment of salaries by the US substituted for a peer enforcement system: when the salary payments were withdrawn we might expect that it would be replaced by a peer enforcement system - but there was no time for this. Hence it was best to surrender right away, and not, for example allow the relatively small number of commandos who were ready to fight to do so. We can of course point to other elements: the corruption of the Afghan government, the removal of air support when the US withdrew and so forth and so on, the fact that those who were called upon to fight - mostly young men - were different from those who benefited from the fighting - mostly women. The point to emphasize is that none of these things are peculiar to Afghanistan. In the case of the Vietnam where the ARVN did fight it is equally true that the government was corrupt and that the US stopped providing air support. Moreover, it is hard to think of any war in which those who fought were not young men a group that received relatively little benefit from victory - that being mostly garnered by the stay-at-homes be they women or older men. More broadly, while support for the Taliban was strong in opium producing areas they were opposed in most other areas, and self-organization in the absence of salary subsidies is not implausible.<sup>7</sup>

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<sup>7</sup>See Mery (2021).

We conclude by pointing out that the theory that the collapse of social norms can produce the “perverse” effect of reducing public good output has support in other realms. In the introduction we pointed to the work of Bano (2012) on NGOs in Pakistan. In a field experiment Gneezy and Rustichini (2000) examined the effect of a fine on the provision of a public bad - picking children up late from a day-care center. They found that this resulted in more parents picking up their children late - an increase of the public bad and the opposite of the expected and intended effect. In a quite different context Dutta, Levine and Modica (2021) show that similar considerations explain why in the face of an enormous drop in demand for oil due to Covid-19 the OPEC+ cartel increased their output. Here also, increased output is a public bad and the exogenous state is not a Pigouvian fine but a reduction in demand. Ordinarily we would expect this to reduce output, but in fact it resulted instead in the inability to reach an agreement over quotas and in a non-cooperative social norm in which output - the public bad - went up.

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## Appendix

Here we prove Theorem 1. The proof follows from a few lemmas proven below.

Recalling that the group solves a mechanism design problem, we denote the optimal choice of  $x$  conditional on  $P > 0$  by  $x^M(\omega)$  and the output of the non-cooperative social norm by  $x^N(\omega)$ . The solution to the design problem  $\hat{x}$  may be either  $x^M$  or  $x^N$ . The corresponding optimal values exclusive of fixed cost are  $u^M(\omega)$  and  $u^N(\omega)$ . We say that  $\omega$  is of *moderate size* if  $(1 + \omega_m)\omega_s \leq c/(1 + \theta c)$ .

Lemma 3 shows that  $x^N(\omega)$  and  $x^M(\omega)$  have strictly positive partial derivatives so are strictly increasing. We also show for moderate  $\omega$  that  $x^M(0) > (1 + \omega_m)x^N(\omega)$ . It follows that the only way in which a reversal can occur -  $(1 + \omega_m)\hat{x}(\omega) < \hat{x}(0)$  - is if the group prefers to use the mechanism  $M$  with punishment at  $\omega = 0$  but reverts to noncooperation at  $\omega$ .

The group will only use  $M$  at  $\omega = 0$  if  $F \leq u^M(0) - u^N(0)$  and will only use  $N$  at  $\omega$  if  $F \geq u^M(\omega) - u^N(\omega)$  and has strict preferences when the inequalities are strict. If  $u^M(\omega) - u^N(\omega) > u^M(0) - u^N(0)$  there can be no such  $F$ . Conversely if  $u^M(\omega) - u^N(\omega) < u^M(0) - u^N(0)$  then there will be a reversal for any  $u^M(\omega) - u^N(\omega) < F < u^M(0) - u^N(0)$ . Hence we must know if  $u^M(\omega) - u^N(\omega)$  increases or decreases with  $\omega$ .

In Lemma 3 we show that at  $\omega = 0$  the partial derivative of  $u^M(\omega) - u^N(\omega)$  is negative with respect to  $\omega_s$  and positive with respect to  $\omega_m$ .

Take  $\omega_s$  first. Since the partial derivative of  $u^M(\omega) - u^N(\omega)$  is negative at zero there is a range of  $\omega_s$  for which  $u^M(\omega) - u^N(\omega)$  is strictly decreasing, hence for any  $\omega_s$  in that range and  $\omega_m = 0$  we have  $u^M(\omega) - u^N(\omega) < u^M(0) - u^N(0)$ . Since  $u^M(\omega)$  and  $u^N(\omega)$  are continuous by Lemmas 1



and 2 it follows that this remains true for  $\omega_m$  sufficiently small. Hence the first result that there is a range of  $F$ 's for which there is a reversal.

Similar reasoning with respect to  $\omega_m$  shows that there is a range of  $\omega_m$  with  $\omega_s = 0$  for which  $u^M(\omega) - u^N(\omega) > u^M(0) - u^N(0)$  and again by continuity this continues to hold for sufficiently small  $\omega_s$  giving the result about no reversal.

The cited lemmas follow.

**Lemma 1.** *The individual optimum is  $x^B(\omega) = \omega_s/c$  with utility  $u(\omega, x, x^B) = \omega_s^2/(2c) + (1 + \omega_m)x - (1 - c)(1/2)((1 + \omega_m)x)^2$ . As the optimum is independent of  $x$  this is also the noncooperative (Nash) social norm:  $x^N(\omega) = \omega_s/c$ , with corresponding welfare*

$$u^N(\omega) = u(\omega, x^N, x^N) = \omega_s^2/(2c) + (1 + \omega_m)\omega_s/c - (1 - c)(1/2)((1 + \omega_m)\omega_s/c)^2.$$

*Proof.* Follows from maximizing the objective  $u(\omega, x, x^i) = -(c/2)(x^i)^2 + \omega_s x^i + (1 + \omega_m)x - (1 - c)(1/2)((1 + \omega_m)x)^2$  with respect to  $x^i$ .  $\square$

**Lemma 2.** *The optimal incentive compatible quota  $x^M(\omega)$  and the corresponding utility  $u^M(\omega)$  are given by*

$$x^M(\omega) = \frac{(1 + \theta)\omega_s + 1 + \omega_m}{(1 + \theta)c + (1 - c)(1 + \omega_m)^2}$$

and

$$u^M(\omega) = \frac{1}{2} \frac{((1 + \theta)\omega_s + 1 + \omega_m)^2}{(1 + \theta)c + (1 - c)(1 + \omega_m)^2} - \theta\omega_s^2/(2c).$$

*Proof.* Given  $\omega$  and a quota  $y = x$ , the incentive constraint is  $u(\omega, x, x) - \pi P \geq u(\omega, x, x^B) - \pi_B P$  so the quota is made incentive compatible by punishment

$$P = [u(\omega, x, x^B) - u(\omega, x, x)] / (\pi_B - \pi).$$

Then monitoring cost is  $\pi P$ , yielding social utility exclusive of fixed cost of

$$\begin{aligned} & u(\omega, x, x) - \theta [u(\omega, x, x^B) - u(\omega, x, x)] \\ &= (\omega_s + 1 + \omega_m)x - ((1 - c)(1 + \omega_m)^2 + c)x^2/2 - \theta [\omega_s^2/(2c) - \omega_s x + cx^2/2] \\ &= (\omega_s + 1 + \omega_m + \theta\omega_s)x - ((1 - c)(1 + \omega_m)^2 + c)x^2/2 - [\theta\omega_s^2/(2c) + \theta cx^2/2] \\ &= (\omega_s + 1 + \omega_m + \theta\omega_s)x - ((1 - c)(1 + \omega_m)^2 + c + \theta c)x^2/2 - \theta\omega_s^2/(2c) \\ &= -(1/2) [(1 - c)(1 + \omega_m)^2 + (1 + \theta)c]x^2 + ((1 + \theta)\omega_s + 1 + \omega_m)x - \theta\omega_s^2/(2c). \end{aligned}$$

Using the fact that maximizing  $Ax - (B/2)(x)^2$  has solution  $x = A/B$  and optimum value  $A^2/(2B)$  yields the values of  $x^M, u^M$  given in the result.  $\square$

We say that  $\omega$  is of *moderate size* if  $(1 + \omega_m)\omega_s \leq c/(1 + \theta c)$ .

**Lemma 3.** *If  $\omega$  is of moderate size then  $x^M(0) > (1 + \omega_m)x^N(\omega)$ ,*

$$\frac{\partial(1 + \omega_m)x^M}{\partial\omega_s}, \quad \frac{\partial(1 + \omega_m)x^M}{\partial\omega_m} > 0$$

and

$$\frac{\partial[u^M - u^N]}{\partial\omega_s} \Big|_{\omega=0} < 0, \quad \frac{\partial[u^M - u^N]}{\partial\omega_m} \Big|_{\omega=0} > 0.$$

*Proof.* From Lemma 2

$$x^M(0) = \frac{1}{(1 + \theta)c + (1 - c)} = \frac{1}{1 + \theta c}.$$

while from Lemma 1  $(1 + \omega_m)x^N(\omega) = (1 + \omega_m)\omega_s/c$ . Hence  $x^M(0) > (1 + \omega_m)x^N(\omega)$  exactly when

$$(1 + \omega_m)\omega_s < \frac{c}{1 + \theta c}$$

which says that  $\omega$  is of moderate size.

Next we assess the partial derivatives of  $(1 + \omega_m)x^M(\omega)$ , where  $x^M(\omega)$  is given in Lemma 2.

We have

$$\frac{\partial(1 + \omega_m)x^M}{\partial\omega_s} = \frac{(1 + \omega_m)(1 + \theta)}{(1 + \theta)c + (1 - c)(1 + \omega_m)^2} > 0.$$

A little algebra shows that

$$\frac{\partial(1 + \omega_m)x^M}{\partial\omega_m} = (1 + \theta) \frac{(1 + \theta)c\omega_s + 2c(1 + \omega_m) - (1 - c)\omega_s(1 + \omega_m)^2}{((1 + \theta)c + (1 - c)(1 + \omega_m)^2)^2}.$$

Divide the numerator by  $1 - c$  and observe that

$$\frac{c}{1 - c} [(1 + \theta)\omega_s + 2(1 + \omega_m)] - \omega_s(1 + \omega_m)^2 \geq \frac{c}{1 - c} [2(1 + \omega_m)] - \omega_s(1 + \omega_m)^2$$

where the right hand side has the same sign as

$$\frac{2c}{1 - c} - \omega_s(1 + \omega_m).$$

So the derivative is positive if  $\omega_s(1 + \omega_m) < 2c/(1 - c)$ ; and this holds by moderation:  $\omega_s(1 + \omega_m) < c/(1 + \theta c) < 2c/(1 - c)$ .

Finally we assess the partial derivatives of  $u^M(\omega) - u^N(\omega)$  at  $\omega = 0$ . From Lemmas 1 and 2 the difference  $u^M(\omega) - u^N(\omega)$  is given by

$$\begin{aligned} & \frac{1}{2} \frac{((1 + \theta)\omega_s + 1 + \omega_m)^2}{(1 + \theta)c + (1 - c)(1 + \omega_m)^2} - \frac{\theta\omega_s^2}{2c} - \frac{\omega_s^2}{2c} - \frac{(1 + \omega_m)\omega_s}{c} + \frac{1}{2}(1 - c) \left( \frac{(1 + \omega_m)\omega_s}{c} \right)^2 \\ &= \frac{1}{2} \frac{((1 + \theta)\omega_s + 1 + \omega_m)^2}{(1 + \theta)c + (1 - c)(1 + \omega_m)^2} - \frac{(1 + \theta)\omega_s^2}{2c} - \frac{(1 + \omega_m)\omega_s}{c} + \frac{1}{2}(1 - c) \frac{(1 + \omega_m)^2\omega_s^2}{c^2}. \end{aligned}$$

From this

$$\begin{aligned} \left. \frac{\partial[u^M - u^N]}{\partial \omega_s} \right|_{\omega=0} &= \frac{(1+\theta)}{(1+\theta)c + (1-c)} - 1/c = \frac{c(1+\theta) - (1+\theta)c - (1-c)}{c[(1+\theta)c + (1-c)]} \\ &= \frac{-(1-c)}{c[(1+\theta)c + (1-c)]} < 0 \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial[u^M - u^N]}{\partial \omega_m} \right|_{\omega=0} &= \frac{1}{2} \cdot \frac{2(1+\omega_m)((1+\theta)c + (1-c)(1+\omega_m)^2) - 2(1+\omega_m)^2((1-c)(1+\omega_m))}{((1+\theta)c + (1-c)(1+\omega_m)^2)^2} \\ &= \frac{(1+\theta)c + (1-c) - (1-c)}{((1+\theta)c + (1-c))^2} = \frac{(1+\theta)c}{((1+\theta)c + (1-c))^2} > 0 \end{aligned}$$

as was to be shown. □