S MODICA
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Representation of Preference Relations

From: Kreps & KLST.

A *binary relation* $\succsim$ on X is a subset of X^2; we write $x \succsim y$ if $(x,y) \in \succsim$. It is *transitive* if $x \succsim y$ & $y \succsim z \Rightarrow x \succsim z$, and *complete* if $\forall x,y \in X$ at least one of $x \succsim y$, $y \succsim x$ holds. A *preference relation* is a transitive, complete (TC) relation (aka Weak Order).[2]

We write $x \sim y$ if $x \succsim y$ & $y \succsim x$, and $x \succ y$ if $x \succsim y$ & not $y \succsim x$. Note that $\sim$ and $\succ$ are binary rel.

A binary relation E on X is an *equivalence relation* if it is reflexive ($x E x$), transitive and *symmetric* ($x E y \Rightarrow y E x$). Let $E(x) = \{y : y E x\}$. [1]

Exercise 1 A family $\{X_i : i \in I\}$ of subsets of X is a *partition* if $\cup_i X_i = X$ and $X_i \cap X_j = \emptyset$ $\forall i \neq j$. Let E be an equivalence relation on X and let $X/E = \{E(x) : x \in X\}$. Show that X/E is a partition of X. $\square$

Starting from pref. rel. $\succsim$, set $\tilde{X} = X/\sim$. Note that letting $\tilde{x} = \sim(x)$, it is $\tilde{X} = \{\tilde{x} : x \in X\}$, and define $\tilde{\succsim}$ on $\tilde{X}$ by setting $\tilde{x} \,\tilde{\succsim}\, \tilde{y}$ iff $x \succsim y$.
This is well defined because $\forall u \in \tilde{x}, v \in \tilde{y}$ it is $u \succsim v \Leftrightarrow x \succsim y$.

An *order* is a pref. rel. R s.t. xRy & $yRx \Rightarrow x = y$.
It is a rank with no ties.

Exercise 2 Show that if $\succsim$ is a pref rel., $\tilde{\succsim}$ is an order.

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[1] Obs. If $\succsim$ is a preference, the rel. $\sim$ is an equiv. rel. In the notation just introduced, $\sim(x) = \{y : y \sim x\}$.

[2] Example: $\geq$ on $\mathbb{R}$.

A binary relation R on X is <u>represented by</u> $v: X \to \mathbb{R}$ if $x R y \Leftrightarrow v(x) \geq v(y)$.

<u>Lemma 1</u> A pref $\succsim$ has a representation if $\tilde{\succsim}$ has one.
Proof. Let $\tilde{u}: \tilde{X} \to \mathbb{R}$ represent $\tilde{\succsim}$; then the $u: X \to \mathbb{R}$ defined by $u(x) = \tilde{u}(\tilde{x})$ represents $\succsim$. $\square$

We want to show that preferences have representations; by ex. 2 $\tilde{\succsim}$ are orders, so if orders have representations, Lemma 1 gives result. We then concentrate on orders in the next three propositions dealing with X finite, countable and uncountable.

<u>Obs</u> For any binary rel $\succsim$ on X, showing $x \succsim y \Leftrightarrow u(x) \geq u(y)$ amounts to showing $x \succsim y \Rightarrow ux \geq uy$ and $x \succ y \Rightarrow ux > uy$.

<u>Prop 1</u> An order on finite X has a representation.
<u>Pf</u> Let $\succsim$ be the order and let $u(x) = \#\{z : x \succsim z\}$ (where $\#A$ = number of elements in A). Then u represents $\succsim$:
if $x \succsim y$ then $y \succsim z \Rightarrow x \succsim z$ so $ux \geq uy$; if $x \succ y$ then $\{z : x \succsim z\} \supseteq \{z : y \succsim z\} \cup \{x\} \supsetneq \{z : y \succsim z\}$ so $ux > uy$. $\square$

<u>Prop 2</u> An order on countable X has a representation.
Pf Let $\succsim$ be the order and $X = \{x_1, x_2, \ldots\}$. Define $u: X \to \mathbb{R}$ inductively: $u(x_1) = 0$, and given u on $\{x_1, \ldots, x_n\}$ define $u(x_{n+1})$ as follows:

- if $x_{n+1} \succ x_k$ $\forall k = 1, \ldots, n$ set $u(x_{n+1}) = n$
- if $x_{n+1} \prec x_k$ $\forall k = 1, \ldots, n$ set $u(x_{n+1}) = -n$
- otw $\exists i, j$ s.t. $x_i \prec x_{n+1} \prec x_j$ and $\forall k = 1, \ldots, n$ $x_k \precsim x_i$ or $x_k \succsim x_j$; set $u(x_{n+1}) = \dfrac{u(x_i) + u(x_j)}{2}$.[1]

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[1] Any finite ordered set has 'best' and 'worst' elements, by induction (as on $\mathbb{R}$ ordered by $\geq$).

Since $X = \bigcup_n \{x_1 \dots x_n\}$ it suffices to show that u represents $\succsim$ on $\{x_1 \dots, x_n\} \forall n$. By induction. True for $n=1$; assume true for n and consider $\{x_1 \dots, x_{n+1}\}$. It is (by induction) $-n < u(x_k) < n \quad \forall k=1,\dots,n$ so u reflects pref. by construction if $x_{n+1} \succ x_k$ or $x_{n+1} \prec x_k \; \forall k=1,\dots,n$; otw if $x_{n+1} \succ x_k$ it is $x_k \preceq x_i$, by ind. hypothesis $u x_k \leq u x_i$, and by construction $u x_{n+1} > u x_i$; analogous if $x_{n+1} \prec x_k$. □

For next prop we need an extra condition. A set $Z \subseteq X$ is dense (in $\succsim$) if $\forall x \succ y \; \exists z \in Z$ s.t. $x \succsim z \succsim y$ (think of $\mathbb{Q}$ in $\mathbb{R}$).

<u>Prop 3</u> An order on uncountable X has a representation if it has a finite or countable dense set $Z \subseteq X$.

<u>Obs</u> this prop. contains the previous two, where $Z = X$.

Pf. Let $Z = \{z_1, \dots z_n, \dots\} \subseteq X$ be dense in the order $\succsim$, and let $r(z_n) = 2^{-n}$. For $x \in X$ define $\overline{Z}(x) = \{z : z \succsim x\}$, $\underline{Z}(x) = \{z : x \succsim z\}$, and u by

$$u(x) = \sum_{z \in \underline{Z}(x)} r(z) - \sum_{z \in \overline{Z}(x)} r(z).$$

If $x \succsim x'$ clearly $\underline{Z}(x) \supseteq \underline{Z}(x')$ and $\overline{Z}(x) \subseteq \overline{Z}(x')$ so $u(x) \geq u(x')$. If $x \succ x'$ $\exists z \in Z$ with $x \succsim z \succsim x'$ and $x \succ z$ or $z \succ x'$ so $\overline{Z}(x) \subsetneq \overline{Z}(x')$ or $\underline{Z}(x) \supsetneq \underline{Z}(x')$ whence $ux > ux'$. □

<u>Lemma 2</u> If a binary relation on X has a representation, it is a preference with a finite or countable dense $Z \subseteq X$.

<u>Proof</u>. Transitivity and completeness follows elementarily using the same properties of $\geq$ on $\mathbb{R}$. For uncountable X,

existence of the dense subset goes as follows: that $u(X)$ contains at most a countable set of 'holes' (intervals $[u(x), u(x')]$ with no image inside); put the extreme of these holes in Z; the rest of $\mathbb{R}$ is filled up, and the counter images of rationals make the rest of Z. $\Box$

Putting together results so far, we obtain

<u>Theorem</u> A binary relation on X has a representation iff it is a preference with a countable dense set.

<u>Example</u> Lexicographic order on $X = \mathbb{R}^2$:

$(x,y) \succsim (x',y')$ if $x > x'$ or $x = x'$ and $y \geq y'$.

Suppose $u: \mathbb{R}^2 \to \mathbb{R}$ represents it. For each x take rational $r(x) \in (u(x,0), u(x,1))$. Then for $x < x'$

$r(x) < u(x,1) < u(x',0) < r(x')$ so r strictly increases, hence establishes 1-1 corresp. between $\mathbb{R}$ and a subset of $\mathbb{Q}$, contradiction.

<u>Exercise 3</u> lexico has no countable dense subset.