WORKSHOPS / ATELIERS

CIEAEM 71

Braga (Portugal)

July, 22 - 26 2019

CONNECTIONS AND UNDERSTANDING IN MATHEMATICS EDUCATION: MAKING SENSE OF A COMPLEX WORLD

CONNEXIONS ET COMPREHENSION DANS L'ENSEIGNEMENT DES MATHEMATIQUES: DONNER UN SENS A UN MONDE COMPLEXE

Odyssey Mathematics: was Homer a mathematician?

Alessio Drivet

GeoGebra Institute of Turin

E-mail: alessio.drivet@tin.it

Abstract: The Odyssey is considered, together with the Iliad, one of the masterpieces of the ancient epic. At a careful reading you can find several references that can stimulate mathematical reasoning. The Odyssey is an epic poem divided into 24 books but only a few books will be considered in the text. Part of the text of the poem contains the reference to the mathematical theme. In the workshop, participants will be able to compare themselves with some of the solutions proposed, especially if they are capable of using GeoGebra.

Résumé : L'Odyssée est considérée, avec l'Iliade, comme l'un des chefs-d'œuvre de l'épopée antique. Lors d'une lecture attentive, vous pouvez trouver plusieurs références qui peuvent stimuler le raisonnement mathématique. L'Odyssée est un poème épique divisé en 24 livres mais seuls quelques livres seront considérés dans le texte. Une partie du texte du poème contient la référence au thème mathématique. Au cours de l'atelier, les participants pourront se comparer à certaines des solutions proposées, surtout s'ils sont capables d'utiliser GeoGebra.

The ancients attributed the Iliad and the Odyssey (along with other poems) to a poet named Homer, a figure perhaps actually lived or, more probably, legendary.
The poem Odyssey consists of three major core themes:
• The Telemachia (books I-IV): four books dedicated to the son of Odysseus, Telemachus.
• Journeys (books V -XII): narrate the sinking of Odysseus following the fury of Poseidon at the Phaeacians and his stay on the island.
• The Return (books XIII - XXIV): here are treated the return of Odysseus to Ithaca and his revenge against the Suitors,

The text has different interpretations, but none takes into consideration the fact that history can be used as a pretext for doing mathematics.
In high schools the work is addressed by Italian and Greek Literature teachers but could serve as an interdisciplinary bridge with Mathematics.
Searching in the poem it is possible to find numerous mathematical ideas that can escape a purely historical-literary reading.
Aim and main idea of the study is the link between mathematics and culture. The methodology is the laboratory method as it integrates the theoretical aspects with the practical ones.

In the Odyssey it is possible to identify some books in which it is possible to imagine a mathematical reference, for the workshop five possible work proposals have been prepared. The project can be interpreted on two different levels: the first is the encounter between literature and mathematics, the second is the use of a useful tool to visualize and solve problems.
The workshop is divided into three parts: the first part briefly presents the references and the consequent proposals, the second part includes the work on the proposed cards (an example is

shown in the appendix), and finally follows the simulations carried out with GeoGebra.
Referring to commonly adopted mathematical terminology, we can speak of algebra, functions, probability, trigonometry and so on.

Book 1
fools, they killed and feasted on the cattle of Lord Hêlios, the Sun

In the first book, the hungry companions of Ulysses eat the oxen of the sun and are therefore punished. The "problem" posed by Homer concerned 7 herds of cows and 7 flocks of sheep, with fifty beasts each. The arithmetic proposed by Homer was trivial and reached a meager number of 700 heads.
Much more interesting is the version attributed to Archimedes. Gotthold Ephraim Lessing (1729-1881), an exponent of German Illuminism, discovered in 1773 the following text:
"Friend, if you participate in wisdom, calculate, using diligence, what was the number of the oxen of the sun that grazed in the plains of the Sicilian Trinacria, divided into four groups of different colors: one as white as milk, the second in black bright, the third tawny and the fourth mottled. In each group there were bulls in quantity, divided according to the following proportion:
1) White bulls = tawny bulls + (1/2 + 1/3) of black bulls.
2) Black bulls = tawny bulls + (1/4 + 1/5) of mottled bulls.
3) Mottled bulls = tawny bulls + (1/6 + 1/7) of white bulls.
4) White cows = (1/3 + 1/4) of all black cattle.
5) Black cows = (1/4 + 1/5) of all mottled cattle.
6) Mottled cows = (1/5 + 1/6) of all tawny cattle.
7) Tawny cows = (1/6 + 1/7) of all white cattle.
Friend, if you really say how many were the cattle of the Sun, what was the number of well-fed bulls and how many were the cows of each color, no one will say that you are ignorant or inexperienced about numbers; yet you will not be numbered among the wise yet. "
The general solution was found in 1880 by A. Amthor. He proved that it was about $7.76 * 10^{206.544}$ cattle.
Here we will limit ourselves to determining the minimum solution of the first part of the problem.
Place a = White bulls, b = Black bulls, c = Mottled bulls, d= Tawny bulls, e = White cows, f = Black cows, g = Mottled cows, h = Tawny cows, we get a system of equations:

$$\begin{cases} a = d + \left(\frac{1}{2}+\frac{1}{3}\right)\cdot b \\ b = d + \left(\frac{1}{4}+\frac{1}{5}\right)\cdot c \\ c = d + \left(\frac{1}{6}+\frac{1}{7}\right)\cdot a \\ e = \left(\frac{1}{3}+\frac{1}{4}\right)\cdot (b+f) \\ f = \left(\frac{1}{4}+\frac{1}{5}\right)\cdot (c+g) \\ g = \left(\frac{1}{5}+\frac{1}{6}\right)\cdot (d+h) \\ h = \left(\frac{1}{6}+\frac{1}{7}\right)\cdot (a+e) \end{cases}$$

The system has endless solutions, to determine a whole solution, it is necessary to assign a value to h, if this value is set to 5,439,213 (minimum common multiple of the denominators) the solution is

obtained:
a = 10,366,482; b = 7,460,514; c = 7,358,060; d = 4,149,387; e = 7,206,360; f = 4,893,246; g = 3,515,820.
Adding these values, we reach the total of 50,389,082 cattle. Indubitably a large herd!

To obtain this solution, GeoGebra can also be used in CAS mode. Once the system has been set, the MCM of the denominators is determined and the desired values are found by substitution (Figure 1) .

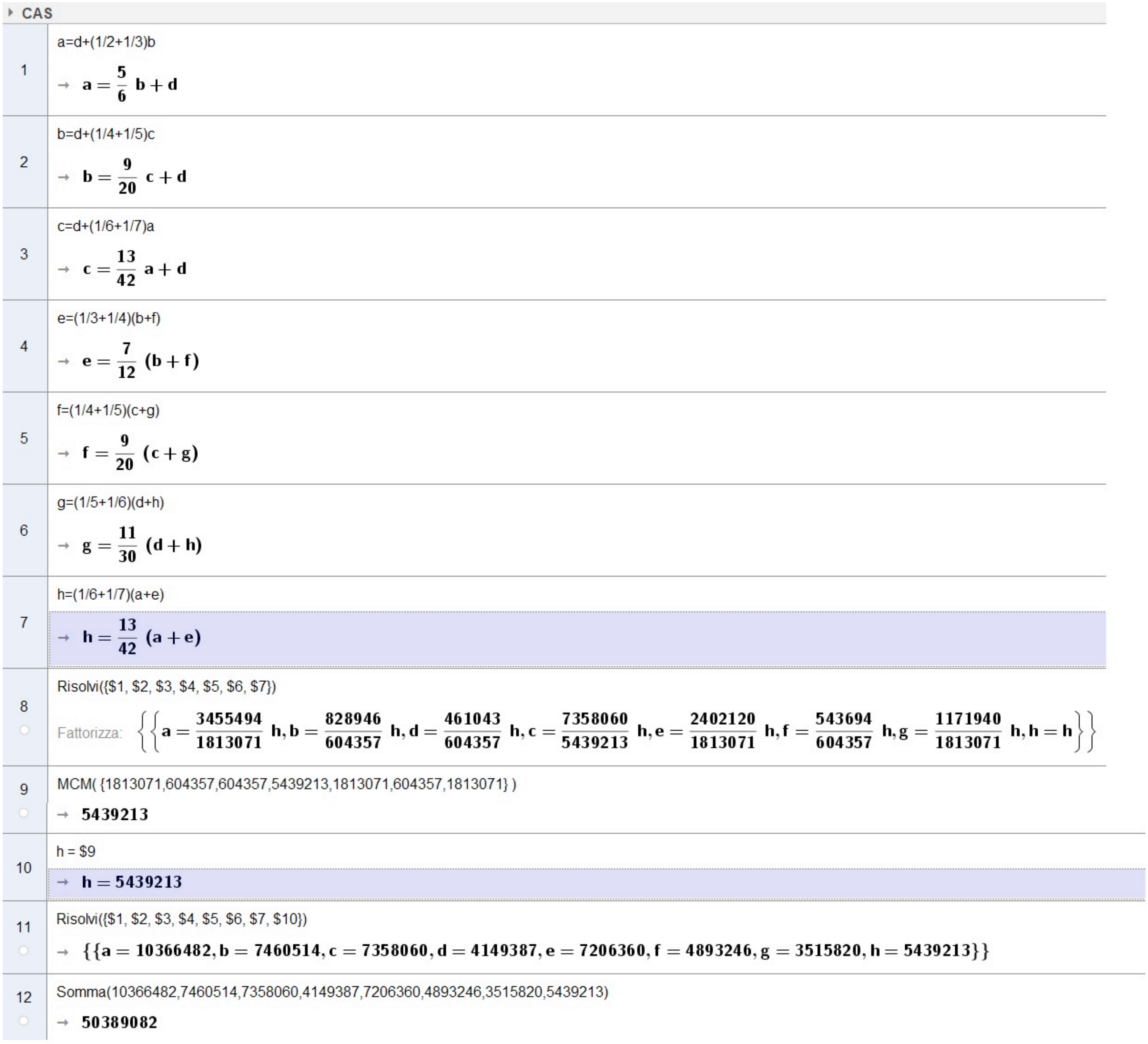

Figure 1. GeoGebra solution

Book 8

One made it shoot up under the shadowing clouds as he leaned backward; bounding high in air the other cut its flight far off the ground — and neither missed a step as the ball soared.

In the story Odysseus, collected on the beach by Nausicaa, the daughter of the king of the Phaeacians, attends the games held in his honor. The story describes a game with the ball between Laodamante and Halios. We could think of some kind of ante-litteram volleyball/handleball.
The ball, launched upwards from a height of x_0 m above the ground, is hit by the hand of the Scratcher at the instant t in which, reached a height of x_1 m from the ground, has assumed zero speed. What is the initial speed of the ball lifter? We know that: $t = v_0 / g$ with $g = 9.8$ gravity acceleration. Therefore, if we replace the previous relation to the equation of motion $s = v_0 \cdot t - 1/2 \cdot g \cdot t^2$, we obtain $v_0 = \sqrt{2sg}$.
To make the solution dynamic, you can use the GeoGebra sliders so we can change the parameters and determine how the speed changes (Figure 2) .

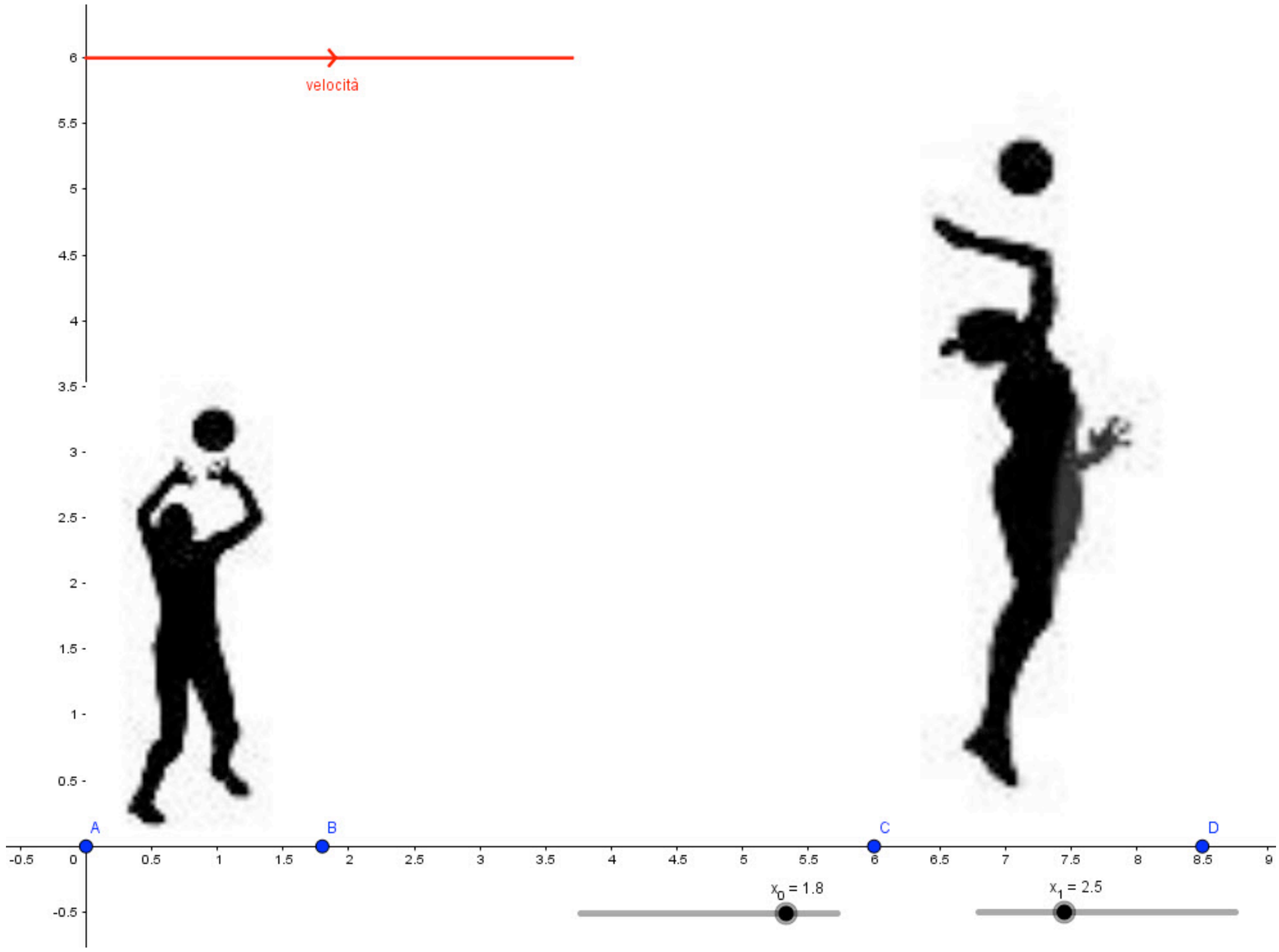

Figure 2. Speed determination

Book 9
The blind thing in his doubled fury broke a hilltop in his hands and heaved it after us. Ahead of our black prow it struck and sank whelmed in a spuming geyser

In the ninth book Polyphemus, blinded, tries to kill Ulysses by throwing large boulders on the ship. A typical, somewhat dramatic, example of the application of probability.
Suppose we represent the marine space as a Cartesian coordinate system, Polyphemus is in the coordinates (0;0). The boat of Odysseus can be placed, quite randomly, in any of the coordinate points (x; y) with x and y between 1 and 6. The blind cyclops throws 3 boulders that can fall into one of the points on the grid where the boat can be found. What is the probability of Polyphemus hitting the king of Ithaca? What if there are 2 or 3 ships?
Let's start from the solution of the simplest problem: what is the probability that launching a boulder Polyphemus will hit the ship of Ulysses?
Obviously the probability will be equal to 1/36, but we can also formulate the problem in a different way by asking ourselves what is the probability of not hitting the ship and then calculating the opposite probability, that is:

$$1-\left(\frac{35}{36}\right)^1$$

As you can easily verify the two values are equal.
The second version, however, is simpler for later calculations. For example, if the boulders are 2, the formula becomes

$$1-\left(\frac{35}{36}\right)^2$$

If there are two ships and three boulders, we can write:

$$1-\left(\frac{34}{36}\right)^3$$

In general, we can say that, indicating with N the ships and M the boulders, you will have:

$$1-\left(\frac{36-N}{36}\right)^M$$

Another way to deal with the problem is to simulate the event a fairly large number of times.
The simulation can be done with GeoGebra assuming a grid in which are placed randomly both the ship and the boulders launched by Polyphemus (Figure 3) .

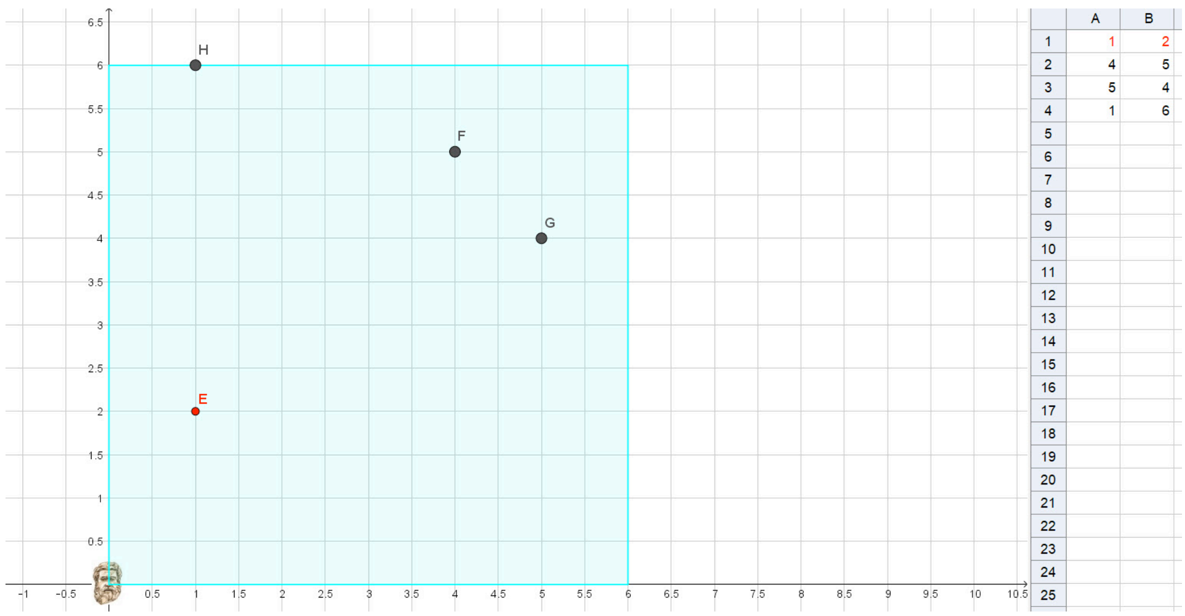

Figure 3. GeoGebra simulation

Book 12

Scylla, where she yaps abominably, a newborn whelp’s cry, though she is huge and monstrous. God or man, no one could look on her in joy.

Book twelve contains the story of the encounter with Scylla, a horrible monster placed on a cliff. The problem can be extended to many other situations, for example the vision of a painting in a museum.

Is there an optimal point where Scylla offers Odysseus' eyes with the widest possible angle?

We call a the height of the rock on which Scylla is located and b the height of the monster.

α is the amplitude of the angle formed by the half-lines that originate from the eyes of Ulysses and the extremes of the rock; β is the amplitude of the angle formed by the half-lines that originate from the eyes of Ulysses and reach the base of the rock and the head of Scylla.

At what distance d should Ulysses be positioned to see Scylla under the angle □ = (□ - □) of maximum width?

In other words, it is a question of maximizing the function f (d).

$$Max\ f(d) = tg\,\gamma = tg(\beta-\alpha) = \frac{tg\,\beta - tg\,\alpha}{1 - tg\,\beta \cdot tg\,\alpha} = \frac{\frac{a+b}{d} - \frac{a}{d}}{1 + \frac{(a+b)}{d} \cdot \frac{a}{d}} = \frac{b}{\frac{d + a \cdot (a+b)}{d}}$$

The maximum of this function, by applying the derivation rules, is

$$d = \sqrt{a(a+b)}$$

The best point is where the eyes are on the circumference passing through the head and feet of Scylla.

In this case we can appreciate the use of GeoGebra for the ease with which we can draw segments and calculate angles (Figure 4).

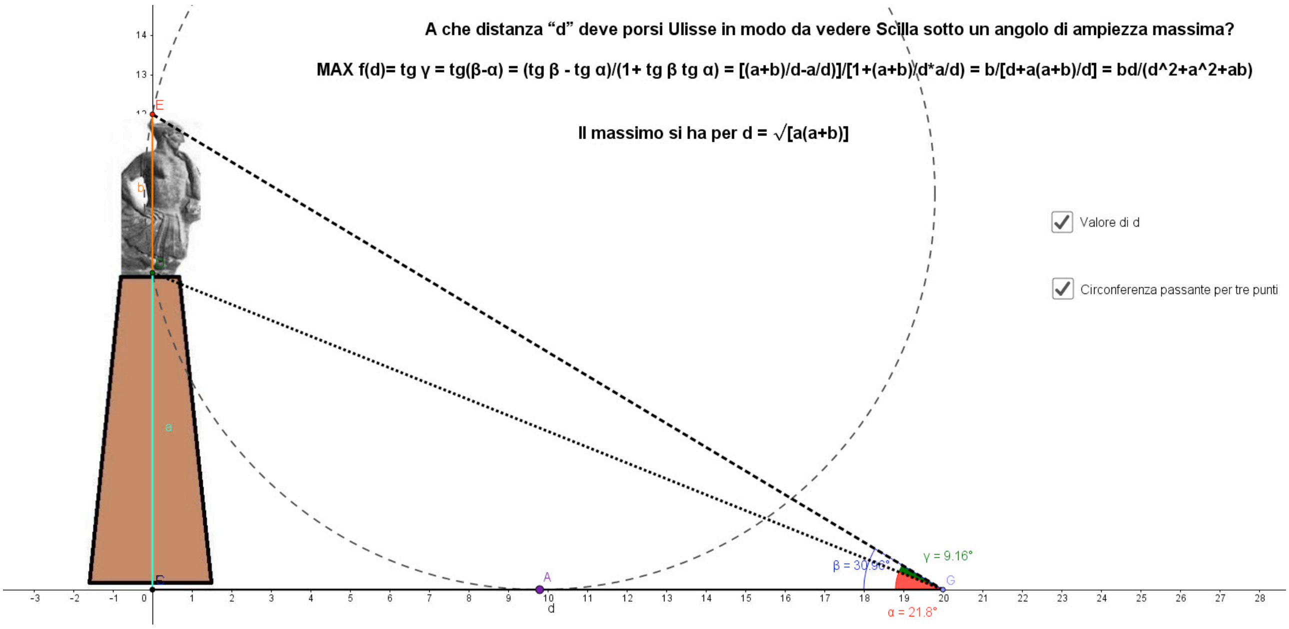

Figure 4. Initial position

Book 21

Now flashed arrow from twanging bow clean as a whistle through every socket ring, and grazed not one, to thud with heavy brazen head beyond.

In a more modern English one could read: The arrow flew from the bow straight through all the holes (of all the axes) and didn't touch any of them. Its tip made out of bronze, hit the ground hard on the other side of the axes.
In the book twenty-one Homer tells how Ulysses tends his bow and passes the arrow in the rings of twelve axes.
The problem is reduced to determining how much the arrow can deviate from its straight path without touching any ring, in other words what is the maximum tolerable angle α. The distinctive parameters of the problem are the radius *r* of the rings and the distance *d* between the arc and the last axe. From here we can determine arctan(r/d) and therefore the value in degrees of the angle α.
The problem does not present any particular difficulties since it is a question of making simple calculations using r = 0,1 m and d = 20 m, for example.
With these data it results that the margin of error is very small as the angle is only 0.286 degrees. The speed with which Ulysses shoots the arrow is just over 68 km / h and the dart takes about a second to complete the route (Figure 5) .

	A	B	C	D	E	F	G
1	r	0,1	m				
2	d	20	m				
3							
4	arctan	0,005					
5							
6	α	0,286	gradi				
7							
8	velocità		$\sqrt{\frac{d \cdot g}{2 \cdot sin\alpha \cdot cos\alpha}}$	19,01	m/s	68,45	Km/h
9							
10							
11	tempo			0,95	s		

Figure 5. Determination of □, V and t

With GeoGebra you can display the angle that leads to the maximum permissible error (Figure 6).

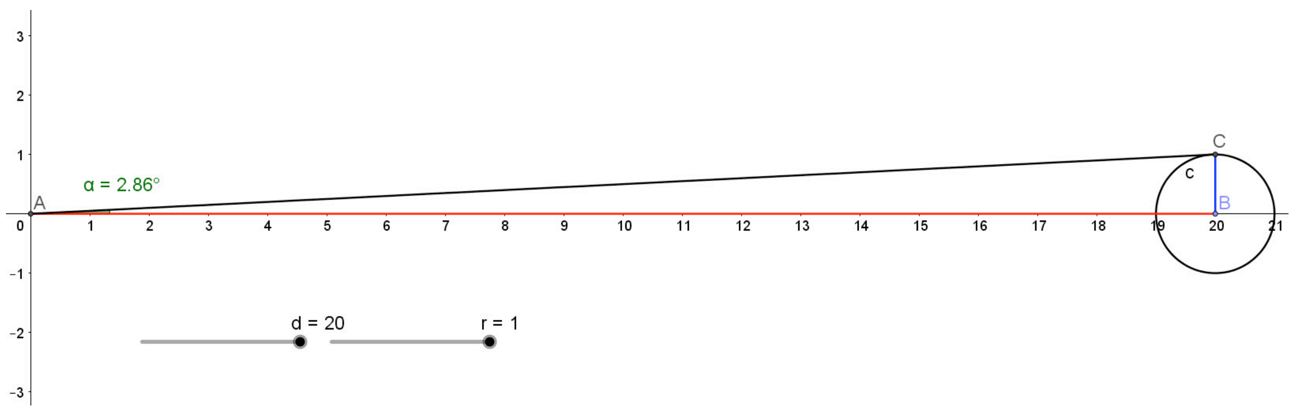

Figure 6. Maximum error angle

Appendix

1 *Fools! who dared to travel the sacred*
In the sun Hyperïon white oxen

Homer's "problem" concerned 7 herds of cows and 7 flocks of sheep, each with 50 beasts. The arithmetic proposed by Homer was trivial and reached a small number of 700 heads.
Much more interesting is the version attributed to Archimedes. Gotthold Ephraim Lessing (1729-1881) writes:
1) White bulls = brown bulls + (1/2 + 1/3) black bulls.
2) Black bulls = brown bulls + (1/4 + 1/5) of mottled bulls.
3) Spotted bulls = brown bulls + (1/6 + 1/7) of white bulls.
4) White cows = (1/3 + 1/4) of all black cattle.
5) Black cows = (1/4 + 1/5) of all skimmed bovine animals.
6) Skived cows = (1/5 + 1/6) of all brown cattle.
7. Brown Swiss cows = (1/6 + 1/7) of all white cattle.
Suppose a = white bulls, b = black bulls, c = mottled bulls, d = brown bulls, e = white cows, f = black cows, g = mottled cows, h = brown cows.

Working proposal: 1) Set the mathematical model (easy):   2) Define the methodological problems related to the solution of the model (less easy):   3) Solve the model with the help of GeoGebra (more or less easy). View CAS write the equations Solve ({\$1..\$7}) MCM({denominators}) h=MCM Resolve({\$1..\$7,\$10}) Sum (values a..h)

References

Drivet A. (2017). *Odissea Matematica*. Independently published. Amazon
Homer. Translated by Butler S., (2011). *The Odyssey of Homer*. Createspace Independent Pub
Homer. Translated by Fitzgerald R. (2007). *The Odyssey*. Vintage Classics
Omero. Translated by Pindemonte I. (2005). *Odissea*. Rusconi
GRIM web site. http://math.unipa.it/~grim/menu_quaderni.htm. Accessed 10/03/2010

Translating practices for reflecting on ourselves: Lesson Study

Carola Manolino, Riccardo Minisola, Ornella Robutti, and Ferdinando Arzarello

Università di Torino

E-mail: carola.manolino@unito.it, riccardo.minisola@unito.it

Abstract. Lesson Study (LS) is a collaborative methodology for teachers' professional development (TPD) rooted in Japan. In LS, a group of teachers and experts collaborates to the detailed planning of a one-hour lesson. The difference between LS and other methodologies is the collaborative foundation of the experience, there is no evaluation on the performance of a single member of the group. We believe that, in the Italian context, LS can be an appropriate tool to answer in an efficient way the Ministry's demands for a "mandatory, permanent and strategic" TPD and for the "establishment of adequate networks for professional collaboration", while maintaining the focus on teachers' needs. Discussing, observing and reflecting on your own and others' practices can help in re-thinking your own professionalism while relating to a community of peers. In the workshop we will work on one of the main tools in LS, the Lesson Plan, and discuss how the encounter of such a foreign tool can lead to self-reflection on one's own practices.

Abstract. Le Lesson Study (LS) est une méthodologie collaborative pour le développement professionnel des enseignants, enracinée en Japon. Dans la méthodologie LS, un groupe d'enseignants et d'experts collabore à la planification détaillée d'une leçon d'une heure. La différence entre la LS et les autres méthodologies est la base collaborative de l'expérience, il n'y a pas d'évaluation sur la performance d'un seul membre du groupe. Nous croyons que, dans le contexte italien, la LS pourrait être un outil approprié pour répondre de manière efficace aux demandes du Ministère pour une développement professionnel des enseignants "obligatoire, permanente et stratégique" et pour "l'établissement de réseaux adéquats de collaboration professionnelle", tout en maintenant une focalisation sur les besoins des enseignants. Discuter, observer et réfléchir sur ses propres pratiques et sur celles des autres peut aider à repenser sonpropre professionnalisme tout en établissant des relations avec une communauté de pairs. Au cours de cet atelier, nous travaillerons sur l'un des principaux outils du LS, le Lesson Plan, et discuterons de la façon dont la rencontre d'un tel outil étranger peut mener à une réflexion personnelle sur ses propres pratiques.

1. Lesson Study

Lesson Study (LS) is a collaborative methodology for teachers' professional development (TPD) rooted in Japan. Since 1999, researchers in TPD and didactics from all over the world have started studying the methodology (Huang & Shimizu, 2016), and since 2003 the Asia-Pacific Economic Cooperation (APEC) has been following its international diffusion. Catherine Lewis, vice-president of WALS, has had an essential role in the world-wide diffusion of LS (Bartolini Bussi & Ramploud, 2018).

In a LS, a group of at least three teachers (which we will call Lesson Study Group; it can include one or more student teachers or university experts) collaborates to the detailed planning of a one-hour lesson, to be taught in one of the teachers' classroom observed by the other teachers, and discussed by the group. The difference between LS and other methodologies is the collaborative foundation of the experience, which leads to the establishment of a sense of diffused responsibility between the members of the group. Moreover, the observation of the lesson indicates LS as a form of action-research. There is no evaluation on the performance of a single member of the group: the focus is on the lesson and the students, not on teachers' individual ability.

The National strategies for Teachers' Professional Development document of the Italian Ministry of Public Education, covering the three-year period 2016 – 2019, stresses the importance of addressing issues such as: teachers' isolation in managing pupils' learning; connecting work and professional development; difficulties in applying in a real-classroom context the didactic innovations proposed by universities. We believe that LS can support the researchers' and teachers' communities in answering the Ministry's demands for a "mandatory, permanent and strategic" TPD and for the "establishment of adequate networks for professional collaboration", while maintaining the TPD focus on teachers' needs. Discussing, observing and reflecting on your own and others' practices can help in re-thinking your own professionalism while relating to a community of peers. The encounter with others, from this point of view, is one's self-rediscovery (Mellone, Ramploud, Di Paola, & Martignone, 2018).

2. Italian Lesson Study: ideas from Turin

As shown from the literature review (Fernandez & Yoshida, 2004; Minisola, 2016; Robutti, et al., 2016), LS is generally a three-step cycle aimed at creating a virtuous process in which teachers can grow continuously (Ramploud & Munarini Frenesi, 2015). In its (cultural) transposition to Western cultures – particularly the Italian one – these steps can be defined the "essentials" of LS: establishment of long-term learning goals and lesson planning, implementation and observation of the research lesson, discussion on the lesson. These steps can be repeated, like a life cycle in which each lesson is the foundation for subsequent growth. In the Italian context, the time expected for each step is: at least 2 hours for goals establishing, 2 hours for lesson planning, 1 hour for the lesson, 2 to 4 hours for the discussion. The overall commitment for teachers is predicted in 7 to 10 hours.

Initial findings mandate to clarify that, in LS, a lesson is a specific moment in the classroom routine (i.e. the mathematics lesson in class 3B held from 9 to 10 a.m. on February the 3rd, 2019). The group of lessons dedicated to a specific topic (i.e. continued fractions) is called teaching unit.

The existence of a LS-Group is tied to that of the observed lesson. Even so, the same group of people can participate in more study cycle and establish a stable-over-time community of practice. The aim is to build and institutionalize a collaborative methodology, which can sustain teachers in both their job and professional development, focusing on the new multicultural context we all are living in.

The tentative structure of Lesson Study in the Italian context is:

Definition of long-term educational objectives: LS is a form of action-research, in which teachers collaborate to improve their professionalism in accordance to the context in which they work. The reasons to engage in LS might stem from different teachers' needs: i.e., difficulties in confronting with certain mathematical topics, improving strategies to involve students, experimenting new didactical methodologies. A research question is formulated by the group in accordance to these needs, and exploring the possible answers is the objective of the LS. Moreover, Italian secondary school teachers have autonomy on defining the educational plan, referring to Indicazioni Nazionali – the national curriculum by Ministry of Education – containing knowledge and competences related to the specific kind of school. Moreover, each school has its Piano Triennale dell'Offerta Formativa or PTOF ("three-year plan of the educational offer"), in which more specific educational objectives are described. Thus, in the first phase of LS in Italy, the teachers choose a teaching unit and the related long-term learning objectives, in accordance to Indicazioni Nazionali and PTOF: these objectives should be relevant to the whole group (i.e. because they are difficult to attain) to promote engagement, and related to the research question(s). One (or two) demonstrating teachers should be chosen, to develop a lesson aimed at a specific context, and to investigate the answer to specific (and yet shared by the group) needs.

Lesson Planning: The demonstrating teacher(s) prepares – on his/her own or working with colleagues – a draft of the Lesson Plan, describing: the class context (such as the general level of knowledge and competencies, or the presence of students with special educational needs: the Italian school is inclusive, meaning that in Italy there are no special schools for students with learning difficulties, physical disabilities or behavioural problems); the teaching unit in which the lesson is inscribed; a proposal for the 1-hour lesson in accordance – as much as possible – to the class' didactic contract. The tentative Lesson Plan is given to the whole LS-Group before the planning meeting, in which the group discusses the details and decides: the phases of the lesson, the time to allocate for each phase, the teacher's requests to students, how the teacher should react to some students' expected reactions, what are the educational aim of each phase, which classroom grouping strategies to apply. The plan is carefully fitted on both the classroom's pupils and the demonstrating teacher's disposition. The group proposes ideas, techniques, strategies, but ultimately it is the demonstrating teachers' choice what to implement and what is not doable in his/her classroom. Observational focuses are established in accordance to the initial aims and to the group's decisions in planning the lesson: i.e. the group might decide to focus on the efficacy of artefacts proposed by the teacher to the class, of the grouping strategies, of the problem structure, etc. The group may decide to use a table to guide the observation using some learning descriptors, and/or to focus on some students considered representative of the classroom situation. In this phase, appointing a secretary to record the discussion, and a moderator to the discussion, is useful in terms of time management.

Lesson implementation and observation: The teacher and the observers enter into the classroom to teach and observe the prepared lesson. A series of preliminary encounters might be necessary to get students accustomed to the presence of other people. The observers are silent and should not influence the class' practice. The presence of all the members of the LS-Group is not necessary, but a video record of the lesson is recommended.

Discussion: LS methodology is focused on the efficacy of the prepared lesson in accordance to the established objectives, not on the ability of the individual teacher. Before the discussion meeting, the whole LS-Group (including the teacher) has shared and studied the observers' reports and possible videos. The discussion is opened by the teacher, who shares his/her impressions and observations on what occurred in the classroom. The whole group discusses how to fix what did not work, improve what did, reflect on how to deal with (and consider the possibility of) the unexpected: it is not possible to plan for every instance that may occur. Missing something is a "mistake" of the whole group, even if the teacher in charge of the lesson was the one who had to respond to the unexpected event(s). As different teachers make different kind of expertise available for the group, this is the opportunity for both the demonstrating teacher and the whole group to learn how to manage unexpected situations or improve non-optimal behaviours, absorbing new ideas from others' experience. This discussion may or may not result in a new "improved" Lesson Plan, which can be taught by the same or another teacher in a different classroom, bringing about a new study cycle. As in the planning phase, choosing a secretary and a moderator is advised.

3. Plan for the workshop

The workshop will be organised in the following steps:

1. A 10 minutes introduction on Lesson Study and the Lesson Plan
2. The participants will be divided in small groups of 3-4 persons each. The groups should be organised as much as possible according to nationality and school level. Each group will have about 10 minutes to decide, possibly according to the participants' training needs, which activity they will work on among the proposed ones (differentiated by school level).

3. 60 minutes will be dedicated to the lesson planning, using the empty Lesson Plan form provided. Each group is asked to keep track of the encountered difficulties, whether they are educational, planning-related, organizational, and also relationship-related difficulties. Since the proposed Lesson Plan structure is adequate to the Italian context, the focus will be especially on those related to the different cultural context the participants are used to.
4. A 10 minutes meta-discussion on the activity: what are the most striking differences between this planning methodology and those used in the participants' usual contexts? Are there any analogies? Does the encounter with the Lesson Plan, a tool coming from a different cultural and institutional context, bring a reflection on the participants' own practices?

4. The Workshop

The workshop lasted 70' and the tentative times were adjusted accordingly: 7' for the introduction, 8' for deciding the teaching material, 45' for lesson planning, 10' for the discussion. 17 people attended the workshop, from a number of nationalities, ages and professional backgrounds. In particular, 7 of them were university students and declared no teaching experience. Audio recording the session was not possible due to GDPR limitations; the presenters noted arguments and comments on a notebook.
During step 1, material on Lesson Study was presented to the attendants. The presentation focused on Lesson Study in relation with the Japanese context, analysed the similarities with the Italian context, and the peculiarities of the latter (i.e.: both contexts focus on the pupils and design educational plans in terms of long-term goals; Japanese teachers mainly work inside schools, Italian ones mainly work at home; Italian school is inclusive, Japanese school is not). A possible adaptation of LS in the Italian context was proposed, and a copy of an empty Lesson Plan (cfr. Appendix) was distributed to each participant. An overview of the proposed structure for the Lesson Plan was discussed, and the participants were proposed the task of the workshop for step 2. The groups resulting by the participants self-organisations were more numerous than expected and heterogeneous, differently from what asked by the presenters. Rearranging them was deemed not necessary. In the case that some of the participants needed support in planning for a lesson, two teaching activities were proposed on different mathematical topics: Heron's problem and its generalization for high school, exploration on non-planar surfaces for middle school.
At the beginning of step 3, the participants were asked to work and discuss within their groups, and three questions were presented to guide the work: (1) What are the most striking differences between this planning methodology and those used in the participants' usual contexts? (2) Are there any analogies? (3) Does the encounter with the Lesson Plan, a tool coming from a different cultural and institutional context, bring a reflection on the participants' own practices? About one third of the participants continued working on the teaching activities; after other 10' the presenter decided to ask to concentrate on the provided Lesson Plan, as it was the focus of the workshop.
By the beginning of the whole-group discussion (step 4), no one had compiled a Lesson Plan: when asked for a reason, a participant commented "analysing and discussing the Lesson Plan is more interesting, as we didn't have enough time to study the material", at which the others agreed. Three participants expressed curiosity about the structure of the lesson proposed in the Lesson Plan, and the presenters explained that it was inspired to Calvani (2014) and Bartolini Bussi et al. (2017).
Participants from France and Spain noted that the philosophy of the Lesson Plan is not far from the work teachers usually do in their context. In Spain, for example, "our teachers do this kind of work for each lesson, and lesson plans are uploaded online for families and others to be consulted"; others from Holland and Switzerland explained that detailed lesson planning is usually mandatory for pre-service teachers during their training but not for in-service ones, and it is usually kept as internal documentation for the school. Two participants from Poland commented that "[detailed

lesson planning] is unusual for both prospective and in-service [teachers], especially with all the details proposed in this Lesson Plan". Spain followed that "estimating the time for each phase, we don't do that" and suggested a modification in the table to highlight this characteristic of the proposed Lesson Plan.
All the participants agreed that specifying the "educational intentionality" for each phase was "the real peculiarity of this document". When asked for clarification, one young participant from Poland explained: "I believe that this is very important when doing pre-service training. It makes you aware that everything you do in classroom is relevant for your students". A participant from Belgium agreed: "it is something you tend to forget even when you are an experienced teacher, so it would be nice to have a reminder every now and then. It makes you aware". The participant from France concluded: "detailed lesson planning is very difficult no matter the subject, but I believe that in Mathematics it is especially important: Mathematics gives students critical thinking, develops their cognitive abilities, is the basis for their scientific approach. We all need to be very careful when we teach it, we are shaping the future… and we need all the help we can get".

5. Discussion

Designing a detailed lesson plan is no easy task, as proven by the fact that no participant produced a complete Lesson Plan, not even those accustomed to detailed documentational work. Furthermore, the task of planning and compiling the plan for a lesson depends on the documentational work that may vary from context to context, for cultural and institutional reasons.
The proposed Lesson Plan is the result of the reflection on the Italian culture, institutional context, usual practices. It embeds Italian institutional peculiarities, such as the different stance on educational objectives: namely, long-term objectives are presented in the National Guidelines, whereas lesson objectives are decided by each teacher individually. A Lesson Plan fitted to a certain context is not immediately effective: its conscious use requires familiarity with the mathematical knowledge, curriculum, teaching traditions, institutional context.
In this sense, we might say that the workshop evolved unexpectedly. The focus shifted from the analysis of Lesson Plan's specific steps to its general issue as a design tool in a school. This new focus nourished the participants' confrontation. The self-organized heterogeneity enriched the discussion, albeit some realism got lost in the transition (namely, no Lesson Plan was compiled), and answered to the needs of the participants.
In conclusion, preparing and studying a detailed plan for an effective Mathematics lesson is perceived both like a challenge and a necessity. Discussing and sharing educational experiences, provided they happen within customary practices of designing and programming, might sustain to collaboratively overcome the perceived challenges of teachers' professionalism. To achieve this, it is necessary that researchers in mathematics education deepen their studies of interaction with teachers in order to improve the collaboration with them in concrete teaching activities.

Appendix

In this appendix, the empty Lesson Plan used in the workshop is presented. Please note that this is a condensed version. The printed version provides enough space to write. This version of the Lesson Plan is inspired by Bartolini Bussi et al. (2017) and Fernandez & Yoshida (2004).

cieaem

Lesson Plan

School:
Class:

Description of the class
(classroom composition;prevalent teaching methodologies)

Context
(learning trajectory in which Lesson Study is held):

Educational goals on competences
(in accordance to the National Programme)

Specific learning goals
(in accordance to the National Programme)

Initial situation of the class
(with respect to mathematical competences: goals already achieved, pre-requisites for the lesson)

Organization of the didactic work
(the total duration of the project; the place where it is carried out)

Organization of the teaching unit:
LESSON 1:
LESSON 2:
LESSON …N:

Methods of evaluation
(how it is intended to evaluate the degree of achievement of the goals, how it is intended to evaluate the functioning of the activity on the class group)

RESEARCH LESSON PLAN

Topic/content of the lesson in question
(title)

cieaem

Goal(s) of the lesson
(Learning goals based on National Programme; punctual and minimum goal of 1h of lesson)

What is the purpose of the observation?
(observational focus)

Presentation of the lesson (mathematics)					
Description of the activity	**Task and/or teacher's questions**	**Student reactions and directions for the teacher**	**Grouping**	**Time table**	**Educational Intentionality (the reasons for the choices)**
Introduction to the lesson and presentation of the topic	*(summary to the class, by the teacher, of the activities already carried out and specific of the topic of the day)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons for doing so)*		
Homework check (optional)					
Formulation/presentation of the problem of the day					*(Explanations of the problematics to be highlighted)*
Presentation / clarification of the problem of the day					
Working on the problem					
Description of the activity	**Task and/or teacher's questions**	**Student reactions and directions for the teacher**	**Grouping**	**Time table**	**Educational Intentionality (the reasons for the choices)**
Working on the sub-problem (optional)	*(if the task is complex, it is possible, intentionally, to break down the problem into simpler units)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons)*		
Working on the problem	*(criteria for group composition; methodologies)*				*(reasons for the choice of activities, materials and methods)*

References

Bartolini Bussi, M. G., & Ramploud, A. (2018). *Il lesson study per la formazione degli insegnanti*. Roma: Carocci.

Bartolini Bussi, M. G., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transposition of Chinese lesson study to Italy: An exploratory study on fractions in a fourth-grade classroom.

International Journal for Lesson and Learning Studies, *6*(4), 380-395. doi: 10.1108/IJLLS-12-2016-0057.

Calvani, A. (2014). *Come fare una lezione efficace*. Roma: Carocci

Fernandez, C., & Yoshida, M. (2004). *Lesson Study: A Japanese approach to improving mathematics teaching and learning*. Mahwah: Lawrence Erlbaum Associates.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher developers, and linking theory and practice through lesson study in mathematics: an international perspective. *ZDM 48*(4), 393-409. Springer.

Mellone, M., Ramploud, A., Di Paola, B., & Martignone, F. (2019). Cultural transposition: Italian didactic experiences inspired by Chinese and Russian perspectives on whole number arithmetic. *ZDM*, *51*(1), 199-212.

Minisola, R. (2016). Insegnanti di matematica che lavorano in collaborazione: panoramica internazionale e contesto italiano. Master's Dissertation. Università di Torino. https://www.researchgate.net/publication/316669730_Insegnanti_di_matematica_che_lavorano_in_collaborazione_panoramica_internazionale_e_contesto_italiano. Accessed 29 September 2019.

Ramploud, A., & Munarini Frenesi, R. (2015, Giugno 10). Il "Lesson Study" [guanmo ke]: trasposizione culturale di una metodologia di formazione. *Scuola Italiana Moderna,* 10, 54–61.

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration: June 2016. *ZDM*, 48(5), 651–690.

ResCo : un dispositif et des situations pour travailler la modélisation mathématique en classe. L'exemple d'un problème industriel d'optimisation de découpes de vitres

Julien Lavolé[1], Simon Modeste[1,2], Sonia Yvain-Prébiski[1,2], et Groupe ResCo de l'IREM de Montpellier[1]

[1] *IREM de Montpellier, Université de Montpellier, Montpellier, France*

[2] *IMAG – CNRS, Université de Montpellier, Montpellier, France*

E-mail: julien.lavole@ac-montpellier.fr, simon.modeste@umontpellier.fr, sonia.yvain1@umontpellier.fr

Abstract. This workshop presents a work developed in 2019 within the ResCo group of IREM (Institute for Research on the Teaching of Mathematics) of Montpellier around the teaching and learning of mathematical modelling. The ResCo group aims to promote teaching and learning modelling in mathematics class and to rely on collaboration between classes to bring students into a modelling activity and make them understand the challenges of modelling. For this, the ResCo group designs problem statements and proposes specific working methods that we will present. We will build on the problem proposed in 2019, "The windows", to illustrate our remarks.

Résumé : Cet atelier présente un travail développé en 2019 au sein du groupe ResCo de l'IREM (Institut de Recherche sur l'Enseignement des Mathématiques) de Montpellier autour de l'enseignement et l'apprentissage de la modélisation mathématique. Le groupe ResCo a pour objectifs de favoriser l'enseignement et l'apprentissage la modélisation en classe de mathématiques et de s'appuyer sur les collaborations entre classes pour faire entrer les élèves dans une activité de modélisation et leur faire saisir les enjeux de la modélisation. Pour cela, le groupe ResCo conçoit des énoncés de problèmes et propose des modalités de travail spécifiques que nous allons présenter. Nous nous appuierons sur le problème proposé en 2019, « Les vitres », pour illustrer nos propos.

1. Introduction

Cet atelier présente un travail développé en 2019 au sein du groupe ResCo de l'IREM (Institut de Recherche sur l'Enseignement des Mathématiques) de Montpellier autour de l'enseignement et l'apprentissage de la modélisation mathématique. Le groupe ResCo a pour objectifs de favoriser l'enseignement et l'apprentissage la modélisation en classe de mathématiques et de s'appuyer sur les collaborations entre classes pour faire entrer les élèves dans une activité de modélisation et leur faire saisir les enjeux de la modélisation. Pour cela, le groupe ResCo conçoit des énoncés de problèmes et propose des modalités de travail spécifiques que nous allons présenter. Nous nous appuierons sur le problème proposé en 2019, « Les vitres », pour illustrer nos propos.

Présentation du groupe ResCo et du dispositif ResCo

Le groupe ResCo (Résolution Collaborative de problèmes) est constitué d'enseignants-chercheurs et d'enseignants du secondaire. C'est un groupe de travail de l'IREM de Montpellier. Il participe à la

formation continue des enseignants de mathématiques. Chaque année, le groupe ResCo élabore un énoncé, appelé « fiction réaliste », en vue de travailler la modélisation mathématique à partir de situations posées hors du champ des mathématiques.
Depuis 2013, les fictions réalistes proposées par ResCo se caractérisent par quatre critères (Ray, 2013) :

- Une situation a priori non mathématique.
- Un contexte fictif mais réaliste.
- La nécessité d'une phase de modélisation pour une prise en charge efficace de la situation.
- La phase de modélisation peut renvoyer à plusieurs problèmes mathématiques selon les choix qui sont faits.

En lien avec le travail de thèse de Yvain-Prébiski (2018) et l'évolution des fictions réalistes proposées, le groupe ResCo a identifié deux critères qui caractérisent ce qui a été qualifié de *Fictions Réalistes Adaptées d'une Problématique Professionnelle de Modélisation* (FRAPPM dans la suite) :
« La fiction réaliste est conçue comme une adaptation d'une problématique de modélisation issue des pratiques scientifiques professionnelles.
Les variables didactiques (Brousseau, 1998) de la fiction réaliste sont choisies de manière à favoriser l'entrée dans la mathématisation » (Yvain & Modeste, 2018).

Le dispositif ResCo

Le dispositif ResCo s'organise de la façon suivante (ResCo, 2014) :

1. Une fiction réaliste élaborée par le groupe ResCo.
2. Plusieurs dizaines de classes, de la sixième (grade 6) à la terminale (grade 12) (Lycées général et professionnel) engagées.
3. Une recherche qui se déroule sur 5 semaines (à raison d'une séance/semaine) selon un calendrier fixé par ResCo.
4. Une plateforme : un forum en ligne, lieu d'échanges des travaux de recherche entre classes (sous la responsabilité de l'enseignant).
5. Des groupes de 3 classes de niveaux proches qui vont échanger tout au long du processus de résolution collaborative.

La recherche collaborative du problème se déroule selon les phases suivantes :
– 1ère semaine - recherche et envoi des questions :
Chaque classe prend connaissance de l'énoncé du problème. Les élèves rédigent des questions pour débuter la résolution. En fin de séance, une mise en commun est effectuée, puis les questions de la classe sont envoyées aux autres classes du groupe.
– 2ème semaine - recherche sur les questions des autres classes et envoi de réponses :
Les élèves répondent aux questions des autres classes. Les élèves les plus avancés dans la résolution peuvent émettre les premières conjectures. En fin de séance, les réponses et réflexions sont envoyées aux autres classes du groupe.
– 3ème semaine - découverte des réponses
Les élèves découvrent les réponses des autres classes, débattent éventuellement de ces réponses.
Entre la deuxième et la troisième semaine, sur la base des questions et réponses déposées sur le forum, le groupe ResCo conçoit une « fiction réaliste relancée ». Elle est adressée à l'ensemble des classes, afin de fixer des choix de modélisation pour permettre la poursuite de la collaboration dans la résolution d'un problème mathématique commun.
Poursuite de la recherche avec la fiction réaliste relancée.
Les élèves découvrent la fiction réaliste relancée et poursuivent leurs recherches sur le problème mathématisé. Les professeurs sont invités à faire prendre conscience aux élèves de la nécessité de faire des choix dans une activité de modélisation mathématique.
– 4ème semaine - poursuite de la recherche :

Les élèves poursuivent la recherche initiée. Les enseignants sont invités à partager les recherches de leurs élèves sur le forum.
– 5ème semaine - clôture de la recherche :
Le groupe ResCo, sur la base des productions déposées sur le forum, élabore et fournit des documents permettant à l'enseignant de clore l'activité. À l'aide de ces documents, le professeur organise un débat scientifique, alimenté par les recherches de sa classe, et celles des autres classes.

2. Présentation de la FRAPPM 2019

La FRAPPM 2019, intitulée « les vitres », que nous présenterons en détail lors de l'atelier est élaborée comme une adaptation d'une problématique de découpe optimale de plaque de verre dans le contexte de la fabrication industrielle de vitres.
L'énoncé est le suivant :

> Une entreprise découpe des vitres rectangulaires de 4 dimensions différentes (210 cm x 215 cm ; 100 cm x 215 cm ; 100 cm x 125 cm ; 60 cm x 215 cm).
> Ces vitres sont découpées dans des grandes plaques rectangulaires de verre de 600 cm x 320 cm. L'entreprise cherche une méthode pour réaliser les découpes selon les commandes en limitant les chutes.
> Pour aider l'entreprise, pouvez-vous proposer une méthode qui réalise les découpes et minimise les pertes?

Critères pris en compte dans la conception de la FRAPPM puis de la version relancée

La FRAPPM « les vitres » respecte les caractéristiques énoncées précédemment d'une fiction réaliste :

- La situation est a priori non mathématique : il s'agit d'un problème industriel.
- Le contexte est fictif mais réaliste : il est inspiré d'un problème industriel réel (adaptation d'une problématique de modélisation issue des pratiques professionnelles - « calepinage ») qui a été légèrement adapté pour supprimer certains éléments de contextes tout en conservant la nécessité de modélisation.
- En effet, il reste nécessaire d'entrer dans une phase de modélisation pour une prise en charge efficace de la situation, en faisant des choix concernant, par exemple, les éléments de contexte pris en compte, la nature des découpes, le sens de « minimiser les pertes » (fonction d'optimisation), ou le fonctionnement des commandes.
- La phase de modélisation peut renvoyer à plusieurs problèmes mathématiques selon les choix faits.

Notre objectif étant l'entrée des élèves dans une activité de modélisation mathématique à partir d'un problème non-mathématisé, nous nous focalisons sur les enjeux de mathématisation sans chercher à identifier les disciplines (scolaires ou académiques) mises en jeu dans le système à modéliser. Cependant, la communication associée à cette atelier (Modeste et Yvain-Prébiski) donne cependant des éléments de réflexion à ce sujet.
Nous présenterons dans l'atelier, les choix de variables didactiques retenues, afin de favoriser le travail de mathématisation en classe.
Sur la base de l'analyse a priori des mathématisations possibles, et de la richesse des problèmes mathématiques qu'elles engendrent, le groupe ResCo analyse les questions et réponses produites par les classes engagées et propose une fiction réaliste relancée qui tient compte de plusieurs critères :

- Le problème mathématisé est accessible, au moins en partie, à tous les niveaux scolaires impliqués

- Les choix de mathématisation effectués conservent une cohérence avec les choix proposés par les classes et ne dénature pas le problème initial.
- Le problème mathématisé permet un travail de recherche mathématique consistant.

Organisation pratique de l'atelier

Dans l'atelier, nous avons fait vivre aux participants le dispositif ResCo en accéléré sur la base de la FRAPPM 2019 « les vitres » (analyse de l'énoncé de la fiction réaliste, phase de questions-réponses, analyse de la fiction réaliste relancée).
Dans un second temps, nous avons proposé aux participants d'étudier des productions d'élèves issues des différentes phases du dispositif. L'analyse de des productions a permis ensuite d'étayer nos propos.

References

Brousseau, G. (1998). Théorie des situations didactiques. La Pensée Sauvage.

Modeste S., Yvain S. (2018) De l'importance de faire des choix. Des situations de « fictions réalistes » et un dispositif de « résolution collaborative » pour faire entrer les élèves dans la mathématisation horizontale [atelier], 69° Conférence de la CIEAEM, Mathématisation, processus social et principe didactique, Berlin.

Ray, B. (2013). les fictions réalistes : un outil pour favoriser la dévolution du processus de modélisation mathématique ? Mémoire de master de l'université de Montpellier.

ResCo (2014) La résolution collaborative de problèmes comme modalité de la démarche d'investigation. Repères IREM 96, p. 73-96.

Yvain-Prébiski, S. (2018) Étude de la transposition à la classe de pratiques de chercheurs en modélisation mathématique dans les sciences du vivant. Analyse des conditions de la dévolution de la mathématisation horizontale aux élèves.

"Quaderni di Ricerca in Didattica (Mathematics)", Numero speciale n. 7, 2020
G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Ethnomathematical study on folk dances: discussing a method for mathematical modelling of choreography

(1) Sara Ribeiro, (2) Pedro Palhares, (3) María Jesús Salinas

(1) CIEC, Institute of Education, University of Minho

(2) CIEC, Institute of Education, University of Minho

(3) Facultad de Ciencias de la Educación, Universidade de Santiago de Compostela

E-mail: sarcristina@hotmail.com

Abstract. This workshop is based on an ethnomathematical study on folk dances. This study aims to analyse the mathematical structure inherent in various elements that constitute folk dances of Northern Portugal and Galicia, specifically choreography, music and accessories (Ribas, 1983). It also aims to construct mathematical tasks related to the ethnomathematical study carried out, linking school-mathematics to mathematics situated in cultural contexts as folk dances. In the workshop, we intend to particularly the method we used for mathematical modelling of choreography, invite participants to learn, apply and experiment the method, and then discuss some of our ideas to apply it in mathematics education classes.

Résumé : Cet atelier est fondé sur une étude ethnomathématique sur les danses folkloriques. Cette étude vise à analyser la structure mathématique inhérente aux divers éléments qui constituent les danses folkloriques du nord du Portugal et de la Galice, en particulier la chorégraphie, la musique et les accessoires (Ribas, 1983). Elle vise également à construire des tâches mathématiques liées à l'étude ethnomathématique réalisée, en reliant les mathématiques scolaires aux mathématiques contextualisées dans des contextes culturels comme les danses folkloriques. Dans l'atelier, nous avons mis l'accent sur la méthode que nous avons utilisée pour la modélisation mathématique de la chorégraphie, et nous invitons les participants à apprendre, appliquer et expérimenter cette méthode, puis de discuter de certaines de nos idées pour les appliquer dans les cours de mathématiques.

1. Introduction

Mathematical activity is a cultural activity (Gerdes, 2007) and mathematical knowledge should be understood as a knowledge that all cultures produce (Bishop, 1988). D'Ambrósio (2002) called 'ethnomathematics' to the mathematics practiced by various cultural groups, which are identified by common objectives and traditions. Bishop (1986, 1988) determined the existence of six universal basic activities (counting, locating, measuring, designing, playing, and explaining) by which mathematics has developed in different cultures and which have contributed to multiple significant ideas that integrate current mathematics. According to Barton (2009), the ethnomathematical study of mathematical practices of particular communities makes us aware of new ideas, concepts and processes that should not be seen as trivial or simple, and may even contribute to new mathematics, capable of enriching the mathematical field.
In today's multicultural society, it is essential to develop a conception of mathematical education that shows it sensitive to social factors, and as a knowledge built on social processes (Moreira, 2008). Ethnomathematics allows educational approaches based on culturally relevant pedagogies

(Rosa & Gavarrete, 2016, p. 26). According to Rosa and Orey (2006), the establishment of cultural connections is a fundamental aspect in the development of new teaching-learning strategies, because it allows students to realize that mathematics is a significant part of their own cultural identity.
These are directions we pursue with our study on folk dances of Northern Portugal and Galicia.

2. Research Problem

The investigation in ethnomathematics we are developing aims to analyse the mathematical structure inherent in various elements that constitute folk dances of Northern Portugal and Galicia, specifically choreography, accessories, and music (Ribas, 1983). Concerning choreography, we intend to represent and describe the movements that dancers perform during dances, as well as the system through which they determine their location and thus organize themselves. Regarding accessories, we expect to identify symmetries that the dancers’ costumes admit. About music, we pretend to recognize repetitive phrases in the musical pieces that accompanies folk dances. This investigation also aims to construct mathematical tasks related to the ethnomathematical study carried out, linking school-mathematics to mathematics situated in cultural contexts as folk dances.

3. Methodology

As we intend to study three elements that constitute folk dances, this is a study with ethnographic characteristics, because it is a descriptive study of the culture of a community or of some of its fundamental aspects (Baztán, 1995), which are, in this case, folk dances. Ethnography attempts to describe the culture or certain aspects of it (Bogdan & Biklen, 1994). Data collection has been carried out in a natural environment through several methods and complemented by information obtained through direct contact of the researcher with this environment (Bogdan & Biklen, 1994).
We studied the dance repertoire of folk groups from two cities of Northern Portugal: Braga *(Grupo Folclórico de Vila Verde)* and Vila Real *(Rancho Folclórico da Casa do Povo de Vilarandelo)*, and from two cities of Galicia: Santiago de Compostela *(Agrupación Folclórica Cantigas e Agarimos)* and Ourense *(Asociación Rebulir Cultura Tradicional)*.
Concerning choreography, video equipment has been used to film under different perspectives the movements that dancers perform on the floor throughout the folk dances. From the video recordings, data has been transcribed (Johnson & Christensen, 2000), and we made diagrams and numerical schemes representing the successive positions that the dancers or the pairs of dancers occupy during dances. Still in relation to choreography, interviews will be used to collect descriptive data in the language of the subjects (Bogdan & Biklen, 1994) on the system through which dancers determine their location and thus organize themselves.

4. Choreography

4.1 Diagrams

To characterize the dynamic process inherent to the choreography, we made diagrams, considering the number of dancers or pairs, the arrangement they assume at the beginning of the dance, and the changes of positions that occur between them.
In the diagrams, the vertices represent the positions of the dancers or the pairs (when the pairs dance together all the time) and they are indicated with capital letter. The directional arcs represent the

movements of the dancers in relation to the previous positions, disregarding the movements that do not include alteration of the relative positions. This means that, for example, dancers’ rotational movements that the dancers or pairs make around themselves are not assumed in the diagrams.
Each movement of the dancers is represented in a diagram. Therefore, the diagrams show how the choreography is performed, based on the initial position of the dancers and the sequence of changes of positions that occurs.
As an example, the analysis of the folk dance “Chula da Ribeiro”, one folk song of the group from Braga, is shown in figure 1. The diagrams represent the sequence of movements that six pairs of dancers (A, B, C, D, and E) do during the dance, starting from their initial positions. Note that the directional arcs that represent the movements of the pair A are marked with a darker color, because if the readers want to focus their attention only on the changes of positions made by one pair, it becomes easier to follow their movements. The other pairs make the same movements.

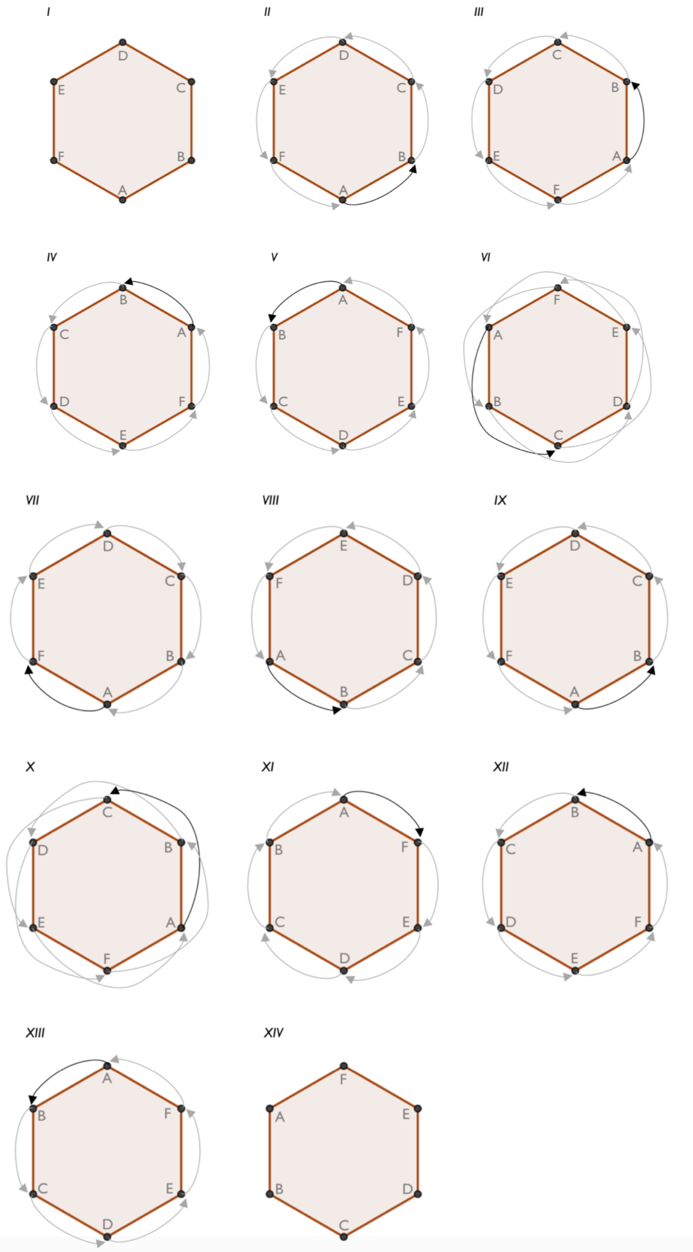

Figure 1 - Diagrams of the analysis of the dance “Chula da Ribeira”.

As another example, the analysis of the folk dance “Regadinho”, another folk song of the group from Braga, is shown in figure 2. Now, the diagrams represent the sequence of movements that fourteen dancers (A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2, G_1, and G_2) do during the dance, starting from their initial positions.

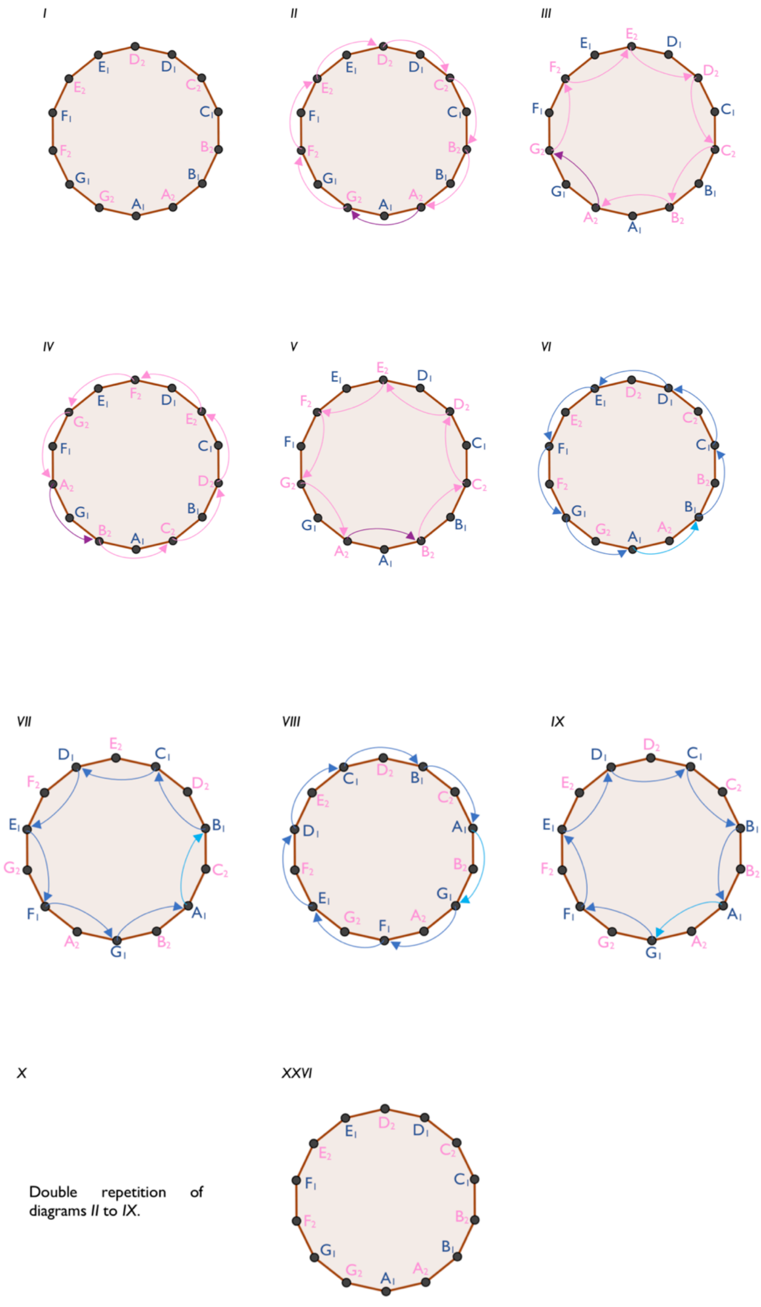

Figure 2 - Diagrams of the analysis of the dance "Regadinho".

4.2 Numerical Schemes

Supported on the diagrams of the analysis of folk dances, we then made numerical schemes. For this purpose, we gave a number to each position of the dancers or the pairs on the initial configuration of the dances (first diagram of dance analysis). Accordingly, the location of pair A (see figure 1) or dancer A_1 (see figure 2) corresponds to position number 1, the location of pair B (see figure 1) or dancer B_2 (see figure 2) corresponds to position number 2, and so on, following an alphabetical order.

For the *circumference choreography type*, in which dancers are arranged in a circumference and their movements are made in this circumference, although sometimes they occupy the area of the circle, we will always use this convention to number the dancers or pairs.
For better understanding, figure 3 shows the position numbers (right side image) assigned to the dancers on the initial configuration of the dance "Chula da Ribeira" (left side image) and figure 4 shows the position numbers (right side image) assigned to the dancers on the initial configuration of the dance "Regadinho".

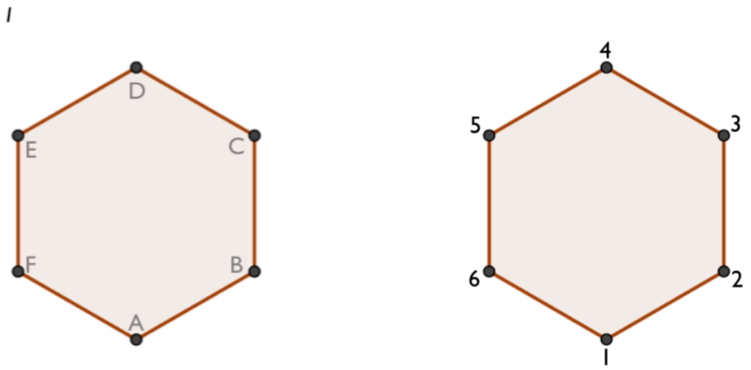

Figure 3 - Numbered positions of the dancers on the initial configuration of the dance "Chula da Ribeira".

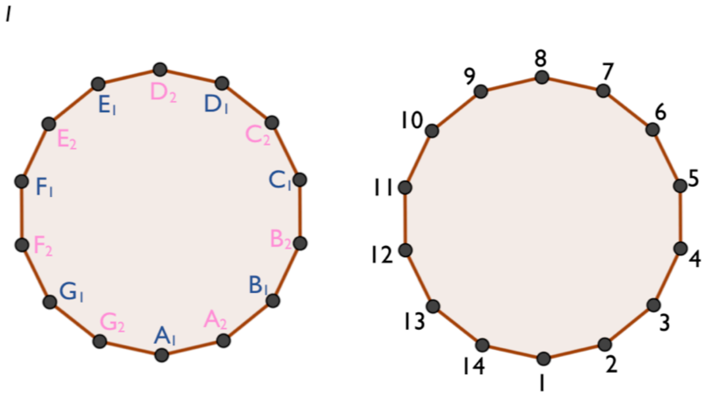

Figure 4 - Numbered positions of the dancers on the initial configuration of the dance "Regadinho".

Based on the position numbers assigned to the dancers on the initial configuration of the folk dances' analysis, the dancers' changes of positions already represented in the diagrams are now described in a numerical scheme.
For each diagram, a row of the numerical scheme is filled, giving the pair A, B, C, D, … or dancer A_1, A_2, B_1, B_2, C_1, C_2, … the appropriate position' number (1, 2, 3, 4, 5, 6, …). This method repeats for all diagrams of the dance analysis.
Figure 5 shows the numerical scheme corresponding to the diagrams of the analysis of the folk dance "Chula da Ribeira" and then figure 6 shows the numerical scheme corresponding to the diagrams of the analysis of the folk dance "Regadinho".

A	B	C	D	E	F
1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
1	2	3	4	5	6
6	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	1
4	5	6	1	2	3
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4

Figure 5 - Numerical scheme of the dance “Chula da Ribeira”.

A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2	E_1	E_2	F_1	F_2	G_1	G_2
1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	12	3	14	5	2	7	4	9	6	11	8	13	10
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
5	2	7	4	9	6	11	8	13	10	1	12	3	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	12	3	14	5	2	7	4	9	6	11	8	13	10
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
5	2	7	4	9	6	11	8	13	10	1	12	3	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	12	3	14	5	2	7	4	9	6	11	8	13	10
1	14	3	2	5	4	7	6	9	8	11	10	13	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
5	2	7	4	9	6	11	8	13	10	1	12	3	14
3	2	5	4	7	6	9	8	11	10	13	12	1	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14

Figure 6 - Numerical scheme of the dance “Regadinho”.

5. The Workshop

The workshop began with a brief presentation of the ethnomathematics research we are developing. Then we focused on the results obtained in the study of choreography. We have described and notated folk dances in terms of dancers’ changes of positions, making diagrams and numerical schemes, which represent the entire sequence of changes of positions that occurs in a dance. As so, we shown how we used the choreography modelling method described in the previous section in one particular folk dance (“Regadinho”).
After this explanatory section, participants were invited to apply the same method in another folk dance (“Chula da Ribeira”).
Our goal was to provide a space for the exchange of ideas, experiences and even cultures, in order to take advantage of cultural practices such as folk dancing to enrich the teaching and learning of mathematics.
During the workshop, participants understood the method and were able to apply it to the other folk dance. In the discussion, several suggestions were made. One of them, perhaps the more relevant, was to count the tempos when the dancers did not exchange positions with each other, which could possibly lead to other regularities. The fact that the method only approached position changes, thus isolating one component of the dance, motivated some discussion about the purpose of the method. Dance is not just the exchanges of positions. Of course not. Who analyzes the schemes does not learn that dance either. Is it useful for choreographers to have the schemes? Is it an aid? What is the purpose of the method if it does not fully portray the dance? These were questions raised, which we reflected on. Regarding the pedagogical potentialities, it was agreed that there are several ways to work on mathematical content through the analysis of the choreography, although no concrete ideas have been suggested.

Acknowledgments

This work was funded by FCT through SFRH/BD/131162/2017 and as part of a project of Research Centre on Child Studies from University of Minho, with the reference UID/CED/00317/2019.
This work was also funded by FEDER/Ministerio de Ciencia, Innovación y Universidades – Agencia Estatal de Investigación/_Proyecto EDU2017-84979-R.

References

Barton, B. (2009). Prefácio. In P. Palhares (Coord.), *Etnomatemática: um olhar sobre a diversidade cultural e a aprendizagem matemática* (pp. 7-10). Vila Nova de Famalicão: Edições Húmus.

Baztán, A. A. (1995). Etnografía. In A. A. Baztán (Ed.), *Etnografía: metodología cualitativa en la investigación sociocultura*l (pp. 3-20). Barcelona: Marcombo.

Bishop, A. J. (1886). Mathematics education as cultural induction. *Nieuwe Wiskrant,* 27-32.

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, 19(2), 179-191.

Bogdan, R., & Biklen, S. (1994). *Investigação Qualitativa em Educação: uma introdução à teoria e aos métodos.* Porto: Porto Editora.

D’Ambrósio, U. (2002). *Etnomatemática: elo entre as tradições e a modernidade* (2a ed.). Belo Horizonte: Autêntica.

Gerdes, P. (2007). *Etnomatemática: reflexões sobre matemática e diversidade cultural*. Vila Nova de Famalicão: Edições Húmus.

Johnson, B., & Christensen, L. (2000). *Educational Research: quantitative and qualitative approaches.* Boston: Allyn & Bacon.

Moreira, D. (2008). Educação matemática para a sociedade multicultural. In P. Palhares (Coord.), *Etnomatemática: um olhar sobre a diversidade cultural e a aprendizagem matemática* (pp. 47-65). Vila Nova de Famalicão: Edições Húmus.

Ribas, T. (1983). *Danças Populares Portuguesas.* Lisboa: Instituto de Cultura e Língua Portuguesa.

Rosa, M., & Gavarrete, M. E. (2016). Polysemic Interactions between Ethnomathematics and Culturally Relevant Pedagogy. In M. Rosa, U. D'Ambrósio, D. C. Orey, L. Shirley, W. V. Alangui, P. Palhares, & M. E. Gavarrete (2016). *Current and Future Perspectives of Ethnomathematics as a Program* (pp. 23-30). Springer International Publishing.

Rosa, M., & Orey, D. C. (2006). Abordagens Atuais do Programa Etnomatemática: delineando um caminho para a ação pedagógica. *Boletim de Educação Matemática,* 19(26), 19-48.