Different representations of functions in a dynamic geometry environment

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Abstract
This presentation relates to a research project at Rutgers University, New Jersey. In an after school program, 13 year-old students were introduced to adding and subtracting integers for identifying rules to match tables for linear and quadratic functions. They worked on their tables to recognize certain properties for linear functions. This work was inspired by the work of Robert B. Davis. The students were given paper and pencil for plotting/drawing and doing calculations. In addition they were using the dynamic geometry software for Casio Classpad 300 calculator. They worked mainly with the PC emulator for the calculator, called ClassPad Manager. The students used the program for plotting points in a coordinate system and for getting the graph and equation of the plotted graph, attending, in particular to properties such as slope, x and y-intercepts, and the domain and range of the functions. In this presentation, the focus is on the development of the students' understanding of linear functions with particular attention to how they made use of certain features of the tool to build and deepen their understanding. The paper will in particular focus on the work of one of the students who presented some convincing arguments about the functions discussed. We will also focus on the possibilities of the tool to promote understanding of the function concept.

Introduction
Early introduction to algebraic ideas is important for all students, particularly for students in urban, economically disadvantaged communities. Failure to learn algebra eliminates students from other courses and blocks opportunities for further mathematical growth. In this paper a description of a program of early algebra learning is presented. The program was implemented with middle-school aged students from an urban public school district as part of an informal math after school program. The program was designed to engage students in algebra investigations based on the work of the late Robert B. Davis in *Discovery in Mathematics* (Davis, 1964). The activities are supplemented by the availability of a technology tool, the Casio ClassPad calculator.

The Project: Early Algebra with Technology
From a larger study with middle school aged minority students in an urban community, the focus is on Charles, a seventh-grade middle-school student who participated in an after-school mathematics program as part of an informal math program. It is reported how Charles built multiple representations of linear algebraic models and recognized the isomorphism between structurally similar tasks, using the technology tool. The enhancement of early algebra learning for middle-school aged students, who have available appropriate technological tools, is an area in need of study.

The Technology: Casio ClassPad 300
Casio ClassPad 300 is a CAS (Computer Algebra System) calculator. The calculator is operated with a pen, as well as the hard keys. The opening screen on the calculator shows a menu, where there are several options for mathematical environments. There is a *Main* screen, where mathematical expressions – such as equations in x and y – can be entered. There are also entries for dynamic geometry, statistics, sequences or 3D graphing to mention a few. For an introduction to the possibilities of ClassPad 300 see (Gjone & Andersen, 2003).

In the dynamic geometry mode it is possible to have a coordinate system on the screen, as well as integer grid points. Working with a grid, points marked will snap to the grid so all the points entered will have integer coefficients. A feature of ClassPad is that one also has the possibility to work in a split screen – dual window – mode, where one part could be the *Main* screen and the other part *Geometry*. Expressions can be dragged from one window to another. One can mark and drag an equation from *Main* to *Geometry* and the graph of the equation will appear. Similarly one can drag a graph from *Geometry* to *Main* and the (equation) expression will appear. Hence ClassPad facilitates working with different representations of mathematical objects. There is emulator software for PCs called ClassPad Manager. Input to the software can be performed with a mouse (simulating use of the pen) using on screen keyboards as well as the computer keyboard. The students in this study were using ClassPad Manager software in the school's computer lab. They were using both the computer mouse and the keyboard.

We integrated the ClassPad tool because of its easy use and its design for students to move freely between graphic, symbolic and table representations. Casio ClassPad, along with Davis' early algebra, provides an approach for students to work with different representations of mathematical objects, such as, linear and quadratic equations. By using the dual window an algebraic representtation of an equation can be dragged to the other window to show a graph, and vice versa. Students hence have the opportunity to experiment with various representations by dragging from one environment to another.

The Students

This study was conducted to investigate how middle-school aged children from an urban low-income minority community learn early algebra ideas, with particular attention to how they translated open-ended problem situations into linear algebraic models using tables, formulas, equations and graphs, with technology software available as a tool.

Eight seventh-grade students, 4 boys and 4 girls, participated in the project. The students were representative of the predominantly black and Hispanic low-income urban community. The students met twice weekly for about an hour and fifteen minutes for seven weeks in the school computer lab. Each session was videotaped with 3–4 cameras. Graduate students took notes and participated together with the research team in debriefing sessions held afterwards. These sessions were also videotaped.

The Davis Approach

The mathematical concepts, activities, and philosophical perspectives that formed the core of the project were based on the work of the late Robert B. Davis (Davis, 1984). The perspective that guides the approach along with a collection of tasks suggests a framework for implementing the lessons. Davis emphasized the use of tools, key ideas, and building representations in designing student activities. He recommended that students work on tasks that elicited opportunities to build meaning and to invent methods of solution. His emphasis on understanding underscored the importance of identifying key ideas and grasping their meaning. For example, Davis identified the idea of variable as a key idea. He suggested that the idea might be viewed as a way of allowing us to say exactly what we plan to do with some number even before we know what number we will actually use. Another key idea is function. He indicated that the key idea for function entailed having some kind of systematic procedure so that, whatever number someone tells us, we will know exactly what to do with it, and therefore be able to respond with the appropriate answer, indicated. According to Davis (1984), by providing students with appropriate experiences in building the key ideas, they would come to think of them in meaningful ways.

Davis presented three underlying themes that suggest a framework for viewing the activities of his Early Algebra Program. He categorized them as: (1) Cognitive Simplicity, (2) Assimilation Paradigms, and (3) Building and Generalizing from Simple Ideas (Alston & Davis, 1996; Maher, 1999).

(1) Cognitive Simplicity

By cognitive simplicity, Davis referred to introducing and using *contexts, notation*, and *language* that are easily accessible and meaningful to children. An example of the idea of cognitive simplicity is the use of a small square (☐), called *box*, and a small triangle ( ), called *triangle*, to represent variables. Davis used the notation in the initial exploration of these ideas with children and adults, before the traditional *x* and *y* notation was introduced. The notion is that the box and the triangle "ask the right question" – namely, that some number is to go in the box or in the triangle. The box and triangle tell you *where the number will go*, but do **not** tell you *what the number will be*. Transition to standard notation occurs as students construct a firm grasp of this key idea.

(2) Assimilation Paradigms

According to Davis, assimilation paradigms are examples in contexts that are familiar and easily understood. Assimilation paradigms could easily be renamed and used as powerful ideas or metaphors for understanding mathematical concepts. The idea builds on Jean Piaget's idea of assimilation.

(3) Building and generalizing from simple ideas

Davis makes reference to situations that present obvious patterns and relationships that can often be modeled with concrete materials with which students can explore and develop mathematical rules. For Davis, the basic notion is that there are many ideas students can invent for themselves in which *telling* would not work. This perspective guided the work in the after-school informal math sessions and is consistent with Davis' view that it is the *student's* responsibility to invent the methods for solving the problems.

Guess My Rule

The *Guess My Rule* activity was introduced by asking students to create a *rule*, using the symbols ☐ and . For example, when one student offered the value 5 for the box, the others applied the rule and suggested the corresponding value, 20, for the triangle. At this point, no one was expected to actually *guess* the rule. A second value, say 0, was given for the box, and the value, 5, was returned for the triangle. A third value, 1, resulted in 8 as the value of the triangle. After a fourth value, 6, for the box resulted in 23 for the triangle, one of the students guessed that the rule was x ☐ . The students who created the rule responded: "No - our rule is ☐ ☐ ☐ ." This provided the context for discussion and agreement about equivalent expressions and the necessity for accepting any of these as essentially the same. The students were then given *truth tables* for a number of functions with the task of figuring out the *rule* that described each table. This activity provided a context for exploring a number of algebraic ideas, including the concept of a linear mathematical function and an introduction to finite difference methods.

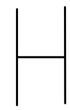
We combined the Davis approach with the dynamic geometry software for Casio Classpad 300. Students had available the Classpad tool for plotting points on a graph and for obtaining the equation of a plotted graph. The Classpad tool made it possible for students to investigate ideas of slope, *x*- and *y*-intercepts, and the domain and range of the function.

We integrated the ClassPad tool because of its easy use and its design for students to move freely between graphic, symbolic and table representations. Casio ClassPad, along with Davis' early algebra, provides an approach for students to work with different representations of mathematical objects, such as, linear and quadratic equations. By using the dual window or screen feature, an algebraic representation of an equation can be dragged to the other window to show a graph, and vice versa. Students have opportunity to experiment with various representations by dragging from one environment to another. This is even possible for higher degree functions. This possibility was not used in this project even though some students were experimenting with quadratic functions.

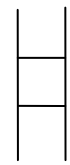
Informal Algebra Learning with Technology

The students had opportunity to explore linear functions during several after school sessions over a seven week period. In the Davis approach to early algebra, students are introduced to *Guess My Rule* activities for identifying rules to match tables for linear functions. They are invited to explore their tables to discover certain properties, such as finite differences for linear functions. Statements of the Ladders and Museum problems:

Building Ladders
A company makes ladders of different heights, from very short ones to very tall ones. The shortest ladder has only one rung, and looks like this: (We could build a model of it with 5 toothpicks.)



A two-rung ladder could be modeled using 8 toothpicks, and looks like this



How could you represent the number of toothpicks needed to "build" a ladder with *any number* of rungs ?

The Museum Problem
A museum gift shop is having a craft sale. The entrance fee is \$2. Once inside, there is a special
discount table where each craft piece costs \$3.
How much would you spend if you bought 10 craft pieces at the discount price?
How much would you spend if you bought 15 craft pieces at the discount price?
How much would you spend if you bought 32 craft pieces at the discount price?
How much would you spend if you bought 100 craft pieces at the discount price?
How could you represent the total amount that you would spend if you were to buy any number of craft pieces at the discount price?

The Ladder problem provided a context that referred to a real object, (building ladders), and offered a concrete material (toothpicks) for constructing a representation of the problem. The students were asked to imagine that the toothpicks were the rungs and sides of a ladder, and that a "ladder company" made ladders of different sizes, from very small ones to very large ones. The two smallest ladders were built: the smallest with only one rung using five toothpicks and the next (two-rung ladder) with eight toothpicks. After building these first two ladders, the students were asked to generate a rule that would give the number of toothpicks necessary for a ladder with any number of rungs.

The Museum Problem was designed by Davis as a final assessment task for the linear function component of his early algebra program. In the Museum problem the students are asked to (a) complete a table of values to show how much it would cost if you buy 1, 2, 3, 4, or 5 craft peices; (b) write a rule using a box ( ) and a triangle (Δ) to show the cost for any number of craft pieces; and (c) make a graph plotting the points from the table created in (a).

Our interest in studying the activity of the students is twofold: first, in examining the representations (graphic, tabular, symbolic) they used to express their solutions and whether they observe connections between the various representations within a particular task and between the Ladders and Museum problem tasks; second, in observing how they used the Casio ClassPad Manager Software in solving problems.

To illustrate the growth of student knowledge in early algebra, we report on the work of Charles, a seventh-grade participant in the after-school, informal math learning study. In particular, we trace the evolution of the development of Charles' understanding of linear functions. We report, also, on his recognizing equivalent representations within a problem and between problem tasks.

The work of Charles
Charles, as well as the other students, was successful in solving both the Ladder and Museum problems. When questioned by a researcher to explain what he did, Charles referred to points on the museum problem graph and found a general rule for the museum problem. He checked the rule using specific instances and made a graph on paper. He then checked the graph by producing a line using the Casio tool. Using the equation the tool produced a linear graph. In a discussion with a researcher about the meaning of the negative values on the horizontal "craft axis" He said that he did not know if he was supposed to use the negative craft pieces, saying, that the negative values didn't "make sense". Charles returned to his paper graph and crossed out the values in the negative domain.

Charles explains his general rule

Charles, while drawing a graph on paper, exclaimed "I did it!" When questioned by a researcher what it was that he did, he answered that he "did his lines". Charles indicated points on the graph in the first quadrant where he drew dots in his paper graph and explained: "If I have one (craft piece) then it is five (total cost). The researcher asked Charles how the graph helped him answer the question and he replied:

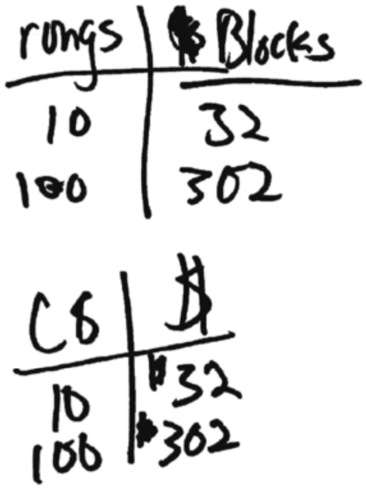
You know that the rule is you multiply the number of craft pieces times 3 and then add 2 because you have to pay the entrance fee and then what you get is how much you spend at that time.

Charles illustrated his rule with another example. He indicated that if you bought 263 craft "sticks" then you would multiply by 3, get 789 and, for the price of admission, add 2.

Charles explains the isomorphism

When asked by a researcher if the problem reminded him of any others he had worked on, Charles referred to the ladder problem. He began by creating a table for a subset of the data of the museum and ladder problems to illustrate the equivalence of domain and range for both problems. Charles explained:

So it is the same thing. [Charles makes a table for rungs and blocks and enters (10,32) and (100, 302)]. This [pointing to the rungs heading] takes place of crafts sticks [he writes CS for the heading for the table below] and this is money [placing $ in the second column and then completing the table]. If you have 10 (pieces), you have 32 dollars; If you have 100 then it is 302 dollars. So it is the same thing. This one [table below] deals with craft pieces and gives you money and this [table above] one deals with rungs and ladders … rungs and blocks.



Charles' representations for the Ladder and Museum problems (above).

Charles further explained how the equations for the museum and the ladders problems were related to each other. He reported that the solution for the Museum problem is "N times 3 plus 2 equals Y." He explained that N stands for the number of craft pieces, Y for the total amount, 3 the price of the craft pieces, and 2 the entrance fee. When asked by the researcher what is the rule for the ladder, Charles responded: "The ladder, which is rungs, could be the same thing as N." He wrote R x 3 + 2 = TB on his paper [he wrote TB for total blocks], saying "three plus two, that is the blocks, total blocks." Then, Charles pointed to the R, drew an arrow, and wrote N. He said "R can take place as the N" and wrote x 3 + 2 =". He pointed to TB, wrote Y after the = sign to complete the equation, and drew a double arrow from TB to Y, saying: "This [TB] can take place as the Y. Charles explained to the researcher that if he did not buy any items in the museum, the cost would be $2.00. He was then asked by the researcher how he built the ladder. Charles responded:

First I just draw the ladder like that. After a while I did it again and again and then I understand and that's how I got the problem.

Discussion

The problems were given as text as shown above. Charles was working with the different representations in the following order:

text → *table* → *rule (using* , Δ on *paper)* → *plotting points (paper)* →
graph (ClassPad) → *equation (ClassPad)*

The graphing paper that was used had dots for the points in the coordinate system, to be similar to what was on ClassPad.

The first four elements in the process were typical *Guess My Rule* activities, which the students had experienced in this project. Plotting two points in the coordinate window (*Geometry*) a straight line through the points can be automatically constructed. This gives the possibility to check if the points are on a straight line, and if there are adjustments necessary.

The technology in this process was only used in a limited way. However, getting the graph on ClassPad by drag and drop highlighted some interesting elements. The graph would extend in both directions and be continuous. Charles entered into a discussion with the researchers of the meaning of certain in values in the domain, specifically negative numbers, and non-integral positive numbers. In this way the technology created an environment where new elements could be discussed.

There are also other possibilities for working with the different representations. The graph in the *Geometry* window can then be dragged to the *Main* window and this will give the equation for the line. The students got a possibility to compare with the rule which they found on paper, seeing the connection between the symbols and Δ, and the *x* and *y* notation.

It is possible to have a more extended use of technology in this context. There is no easy access to tables in the *Geometry* environment on ClassPad. However, ClassPad has a built in spreadsheet where a function table is an obvious element. In a spreadsheet one can moreover have different forms of graphical representations, e.g. point-graph or line-graphs. This could lead to more extensive experimentations.

The students who participated in the project mastered the technology easily, and needed almost no instruction. An extension of this project could therefore include more advanced computer use.

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