Concentration or Lack of it

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Abstract
There is pedagogical evidence that in order to fully understand any abstract notion, several different embodiments or incarnations of that concept must be considered, and the learner must not only be aware of them, but also be confident enough to juggle them and move between them according to the context. In this study I intend to analyze how multiple representations support a better and deeper understanding of mathematical subjects. It is about gaining the flexibility of moving from one representation to another as one aspect of knowledge and understanding. This ability to recognize one concept in different forms and representations broadens and deepens knowledge and allows connections between notions and concepts, thus strengthening problem solving skills. (Even, 1998)

Introduction
A representation is defined as any configuration of characters, images or concrete objects that can symbolize or represent something else (Kaput, 1985; Goldin, 1998 and Dewindt-King). Modes of mathematical representations may involve words, graphs, tables, and equations. Symbol manipulation skills include being able to carry out arithmetic and algebraic procedures. In this paper I promote raising awareness about the role of multiple representations and the designing of supporting learning material that provide students with coordinated, multiple representations, one of which is coordinated visual and verbal representations (Kozma, 2003), all for the purpose of enlightenment and evolution of the students cognitive skills. Learning is characterized as becoming attuned to learning activities that result from interactions between the students and their material and representational resources as they engage in inquiry (Kozma, 2003). "The presentation of information in both visual and verbal forms increases recall and problem solving transfer by helping learners encode this information in both visual and verbal forms and integrate these forms in long-term memory" (Janvier al., 1987). There is enough evidence in science education on the effect of multiple representations on mature understanding: for instance experts are able to group a cluster of diverse problems or situations into a large meaningful group based on underlying principles, using conceptual terms to label their clusters by resorting to a greater variety of representations; while at the same time, and for the same cluster, novices or students would organize that group based on "surface" features, using terms that merely describe the surface features of the groups (Kozma, 2003). It is known by empirical research that the student's construction of a mathematical object is based on the use of several semiotic representations (Hitt, 1998). The learner's handling of different mathematical representations will permit ways of constructing mental images in the sense of Vinner and Tall. By concept image, Tall and Vinner (1981) refer to something non- verbal associated in our mind with the concept name: it can be a visual representation or even a collection of impressions or experiences, which may be translated into verbal forms. Many times the richness of the learners' concept image will depend on their handling of the representations used.
The inclination to remain restricted to one representation system (say algebraic and symbolic) might produce errors in problem solving situations. According to Aspinwall (Aspinwall et al., 1997) "symbol manipulation has been over emphasized, and in the process the spirit of calculus has been lost". One wonders why individuals in general get attached to the algebraic system of representation. And why do students avoid visual reasoning? It could be partially due to the fact that teachers continue emphasizing in their instruction on non- visual methods.

In general, instructors intensively use algebraic representations in order to avoid confusion between mathematical objects and their representations (Hitt, 1998). However, by doing so, they neglect or under-emphasize the geometric and intuitive or global representations. For some reason, people in general tend to assume that algebraic systems are more formal than other non symbolic representations;

Literature Review

Recent research indicates that the ways in which students think about mathematical concepts in general may be surprisingly different from what we might expect (Ferrini et. al, 1993). Techniques for understanding mathematical representations are rarely directly covered in class, and lack of this understanding underlies many of the misconceptions that impede student progress in subsequent classes (Brenner et. al., 1997). It is a fact that there is a lot of research on the topic of theoretical and practical translation between and within representations (Goldin, 1987, Janvier 1987, Moschkovich, Schoenfeld and Arcavi, 1993, Monk, 1988). Kaput, Janvier (Janvier et. Al, 1987) distinguished between signifier and signified: the internal representation is the signified, which can be illusive since such an internal representation cannot be directly observed. Whereas the external representation is the signifier which acts as stimulus on the senses such as the case with computer graphics or charts, or in general any embodiment of ideas. Usually the relation between the external and internal representations is expressed in the form: "external" means or signifies "internal". According to Goldon (15), it is a system of coordinates of the form (external, internal) paired up according to a given rule. That set of rules for organizing the external representation is referred to as Syntax. However the way the interpretation is assigned belongs to Semantics. A shape of a graph supports the semantics of the representation. Mason (1987) suggests that when designing a lesson, an instructor would benefit from a spiral movement based on the distinction between iconic and internal representations of a concept. At any stage, a question may induce movement back down the spiral to restore confidence. The distinction between iconic and non-iconic representation was first used by Bruner (1966), who was considered a pioneer in the study and uses of representations in pedagogy. A common distinction today is drawn between the operational approach to a concept as a process, and a structural approach as an object. In many cases in calculus the distinction is made between the global/qualitative and the pointwise/local approach. When it comes to functions, the pointwise approach is used for plotting, reading or dealing with discrete points on the graph, or when one is interested in studying the behavior of a function over a specific set of points. It is often observed that translating a functional relationship between data pairs into algebraic symbols is one of the very difficult tasks for students (Kieran, 1993).

Illustrative cases of handling multiple representations

The way in which instructors convey the material sets the tone of how students would react to it and defines the distance between the learner and the new material. Students can be indirectly or directly trained to look for indicators that suggest the appropriate representation to be used in a certain context.

> **Polar versus Cartesian coordinates**: Calculus textbooks in general do not motivate enough the introduction of polar coordinates, which leaves students puzzled in face of this bizarre way of giving addresses to points in the plane (or space); according to them life was going fine with the rectangular way of locating points. One way I handled this dilemma was to introduce in a dry manner those new polar coordinates and even do a little practice and applications, and then ask the students themselves about what they think could have been the source of this sudden need for new types of coordinates. This is

some of what I heard: "While the Cartesian coordinates reflect the erroneous flatness of the earth (or plane), polar coordinates suggest that the earth is round". Also "since we perceive and see through our eyes, and since the iris of the eye seems to be divided into orbits and rays the closest to our vision, this configuration can be easily described by polar coordinates, (r=constant) for the orbits and (*theta*=constant) for the rays. Long discussions follow as to what shapes and graphs are more readily expressed by polar coordinates and which figures call for the Cartesian coordinates. Such questions raise awareness about the multiplicity of the representation of points and sets of points in the plane, and automatically place the learner at the higher level of a "Chooser". Of course such discussions and debates will only be meaningful and useful if they follow enough drilling on routine exercises that secure the required confidence with working with the new coordinate system and even with translating between the two systems. Another instance of choosing between polar and Cartesian is in complex variables, when students have the option of referring to the complex number as $z=a+ib$, or $z=re$^(*itheta*). They can be also be asked to choose the appropriate representation depending on the context. They always come up with the right answers when properly directed and guided: they'd rather go for the polar form in case they are interested in multiplication, in raising to a power, or in finding the roots of the unity (while solving for z in z^$n=1$), while they would always resort to the Cartesian representation when the problem involves say, addition. Also when they want to determine the image of a certain region in the domain under a complex function, their choice of the representation would depend both on the function and the domain of integration. A similar choice has to be made when evaluating a double integral: the nature of the integrand and the domain of integration determine the appropriate form to be used. If a change of representation needs to be performed, it must be preceded by a sketch of the domain of integration. Such a sketch is also required for determining the ideal order of the variables (*dxdy* or *dydx*). Note that this requirement (of a sketch) is also an evidence of how it is indispensable to switch sometimes from the symbolic manipulation (of integration) into its geometric representation.
The "iconic" integral sign: Instructors should never miss a chance to solicit students' comments around the adoption of new (mathematical) symbols: in one of my calculus classes, a student once commented on the (iconic) representation of the symbol of integration (after I had introduced the formal Riemann sums followed by integrals and the fundamental theorems of Calculus) by noticing that the slanted S symbol seems like a "curved, continuous" metamorphosis or evolution of the "discrete pointwise sum" presented by the sigma symbol of summation (of the areas of the rectangles). This sharp comment reveals that the student does not perceive the icon as an alien symbol imposed by some higher authority, but rather as a choice, and somehow suggests that the student participated in its creation.

Proof writing: wise choice of external representations as "typical elements": when one intends to show set inclusion (A subset of B), the key to a successful naturally flowing proof falls many times in a wise choice of the form of the *typical* element of A (that needs to be shown also element of B). This clever preference of the typical element may guarantee a successful proof. In Abstract Algebra for instance, in order to show that the coset aH is included in the coset bH, students who start the proof by writing "let ah be an element of aH for some h in H" have a better chance at completing the proof than those who start by writing, say "let x be an element in aH." The iconic representation ah

is key to the proof; it is as if by doing so, that student implicitly acknowledges the definition of the coset aH.

Multiple representations of derivatives: In traditional outdated textbooks it is introduced as the limit of that ratio $\Delta y/\Delta x$ as Δx tends to zero (rate of change). In physics, it is introduced as a velocity; graphically, it is the slope of the tangent line to the curve at a given point; and technically it can be calculated by applying a set of rules and techniques, as in the addition rule, quotient rule, chain rule, and so on. A typical calculus student is not always able to surf between these representations and to recognize that all these forms are equivalent. Same for the case of the different forms of the partial derivative: in order to verify that students have firmly established the desired links between the various representations of the partial (or directional) derivative, the instructor, and after the conventional traditional introduction and the drilling exercises can ask the students to find say the slope of the line tangent at the point $(2, 3, f(2,3))$ to the intersecting curve between the surface $z=f(x,y)$ and the plane $x=2$. Having the correct answer (f_y (2,3)) is a sign of a deep understanding of the partial derivative. This exercise is equivalent to asking a person for that (dictionary) term corresponding to a certain meaning as it appears in a dictionary, rather than ask them for the meaning of the term itself. In other words, it is equivalent to asking for the translation between two representations in the unusual direction.

Set Theory: proofs with Venn Diagrams versus symbol driven proofs: what is more natural, and what is more rigorous? Some teachers do not consider a Venn diagram proof a real proof, although students like to resort to this proof and find it simple and meaningful. Those teachers would rather see symbols in rigorous template-like statements of the form "let *x* be an element of … Let us show that *x* is also and element of …" However, students find Venn diagrams quite convincing because the visual representation of sets appeals to them. As with any other type of proof, there is always a chance that students may be applying a template without understanding why it works, or what makes it work. I have some reservation against the abuse of those Diagram proofs. If we consider the case of De Morgan's Law, ~(AUB)=(~A)∩(~B). One would assume that students using the Venn Diagram proof are aware that the universal set is being partitioned by the two sets A and B into the four (non-overlapping) sets A-B, B-A, A∩B, and finally ~ (AUB), for only in case they are should this proof be accepted. But in general instructors fail to diagnose such an understanding or to even highlight it enough in their lectures. (Strange enough that in most textbooks the subject of a partition comes later in the course). Same for the case of the inclusion-exclusion principle, which is directly verified by Venn diagrams, at least for the case of two or three sets. One wonders if all the diagram or visual proofs hide a certain assumption that can be abused or overlooked.

Conclusion

Once learners are made aware of the different representations of one concept, the next step would be to make a choice about which representation is the most appropriate and meaningful at a given situation. In APOS terms (Dubinski, 1989), making a choice will be the "process", whereas the different representations would be the tools, or "objects".

On the other hand, some researchers argue that adding more representations and personifications to a concept does not guarantee necessarily more meaningful internal representation, and may keep the process as a whole at the surface level. A balance would naturally be the best. Students should be allowed to explore any new representational system until it becomes meaningful to them personally. Instructors always strive for that magic balance between the multiple

embodiments of a concept and the need for connections between those meanings and those external representations.
Finally, as an evidence of the indispensability of moving between representation systems to solve certain problems, what is better than the role of Analysis in the proof of the fundamental theorem of Algebra, the two A fields that otherwise seem parallel.

Acknowledgment: This study has been partially funded by the Lebanese American University.

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