A non-standard course for future High School mathematics teachers

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Abstract
A course on teaching secondary mathematics can help to bridge the gap between the mathematics usually taught at college level and the mathematics prospective teachers will need to teach in High School. The content and guiding principles are important factors to the success of such a course, as well as the proper use of technology and a close relationship with physics.

Introduction

At most colleges and universities, future High School teachers of mathematics have to earn a B.S. in mathematics and take several education courses. East Tennessee State University is no exception to this arrangement. The department of mathematics at ETSU offers the mathematics education track, which requires passing sixteen mathematics courses (calculus, abstract algebra, real analysis, differential equations, college geometry, and the like) as well as Teaching Secondary Mathematics (Math 4417).

Although the above-mentioned sixteen courses provide an overall good mathematical background, there is a certain lack of material directly relevant to High School mathematics. To remedy this situation we are converting Math 4417, which currently has college geometry and abstract algebra as prerequisites, in a sort of capstone course where we cover several topics from three broad areas, namely algebra, geometry, and plane trigonometry. The advisability of a capstone course is widely recognized (CBMS, 2000; Hill and Senk, 2004). Our two guiding principles revolve around presenting multiple perspectives to the solution of some problems and the genetic approach to teaching mathematics whenever this is possible.

Content

Students have the opportunity to study roots of cubics, biquadratics and quartics, tangents to conics through a modification of a method originally invented by Descartes (Baloglou and Helfgott, 2004), maxima and minima problems (Natanson, 1963), as well as the addition formulas of trigonometry and their relationship to Ptolemy's theorem (Gelfand and Saul, 2001). All these topics, and many others, belong to the field of precalculus mathematics; nonetheless they require mathematical maturity.

Although we do not use any calculus techniques, the idea of limit of a sequence does appear quite often. For instance, we deal with such problems as the derivation of the volume of a cone and a sphere, the area of a segment of a parabola, and diverse approaches to the approximation of π . A detailed list of topics appears at the end.

One of our principal goals is to show the connections between the mathematics students learned in their first three years of college and High School mathematics. The existing disconnect has to be addressed before future teachers graduate (Cuoco, 2001). Besides, we must foster creativity among students. They should learn that the creative process usually precedes the logical development of a proof. Is there a textbook that is compatible with our vision? Unfortunately not, although some noteworthy books have been published in the recent past (Usiskin et al., 2003; Cuoco 2005).

Guiding Principles

To show the richness of mathematics, here and there we present alternative approaches to the solution of problems. Geometry is a fertile ground, wherein either the Euclidean or Cartesian framework can be employed (Taback 1990). On the other hand, a surprising number of optimization problems can be tackled with such non-calculus techniques as the arithmetic-geometric mean inequality. Of course, when the function being considered happens to be a quadratic or biquadratic there is a formula to find where it attains its maximum or minimum.
Our second guiding principle has to do with the genetic method (Mosvold, 2003). That is to say, we try to reach the genesis, the origin, of a mathematical idea in order to shed light on the connections with other concepts and put matters in historical perspective. The search of roots of polynomials, especially of third and fourth degree, is a case in point because it illustrates the role played by Cardano and Ferrari, and provides the opportunity to present the subject from multiple perspectives.

Applications

Applications to physics play an important role in Math 4417, especially to mechanics and optics. We stress our conviction that the use of applications in the K-16 classroom setting is an important pedagogical tool. For instance, elementary kinematics is a good option (Newburgh, 1996), or how the range of an object can be analyzed under the absence of resistance of air -- an idealized situation -- or the more realistic setting when resistance of air is considered. For the latter case the Lambert W function is of great help (Packel and Yuen, 2004). The depth of a well, when the speed of sound is taken into account, is another example where elementary kinematics illustrates a mathematical concept (Polya, 1977). Optics is a good source of problems too, be it the phenomenon of reflection on plane or parabolic mirrors, or the phenomenon of refraction. Huygens purely geometrical approach to refraction, in its modern formulation (Golomb, 1964), is accessible to students that have taken trigonometry only, while the discussion of Newton's parabolic telescope is a nice opportunity to show the interplay between mathematics and physics, as well as to illustrate the close relationship between Euclidean and Cartesian methods. We do cover in class the physical ideas behind the above-mentioned examples, taking into consideration that not all students have a good background in physics. Although we believe in some sort of integration between mathematics teaching and science teaching, we do keep in mind all the pitfalls that have to be avoided in this process of integration (Steen 1994).

Pedagogy

Teaching Secondary Mathematics is primarily a content-based course. However, we have to provide students with some specific knowledge related to methods, namely the ability to write coherent and realistic lesson plans (Posamentier and Stepelman, 2006) and to deliver them appropriately. It is to be noted that at the beginning of the semester students have to take a competency test to demonstrate proficiency in High School mathematics (Algebra II, Geometry, and Trigonometry). Such an arrangement gives us an idea on how well prepared they are before going into teaching, and there is always the opportunity in Math 4417 to fill any gaps in their knowledge.
Polya's ten commandments of teaching (Polya, 1962) are always at the forefront of all we do with regard to mathematical pedagogy. Our emphasis on content rather than methods of teaching is in consonance with the need to reach a balance in the preparation of future teachers of mathematics (Wu, 1997).

Technology

We are thinking about graphics calculators (mostly TI-83 or their equivalents) and computers whenever we deal with technology; more the former than the latter. Students learn to write their own programs to enhance the capabilities of their calculators, not as a theoretical pursuit but as a means of approximating π, calculating the probabilities behind the birthday problem, dealing with diverse recursive algorithms, and testing conjectures. Moreover, we stress the fact that a quasi-empirical approach to certain topics is possible with a judicious use of graphics calculators and geometry software (Helfgott, 1998). As expected, the use of technology is intended to supplement, not to supplant, mathematical learning. An example may better convey some of our ideas about the use of calculators to test conjectures.

Suppose that we have a rectangular closed box whose base is a square. The surface area of the box is a given number L. What are the dimensions of the box of maximum volume? Let x be the side of the base and y the height of the box. Then $4xy + 2x^2 = L$, so $y = (L - 2x^2)/4x$. Thus

$$V(x) = \frac{x}{4}(L - 2x^2)$$

Next we choose a value for L, say $L = 5$, and use our graphics calculator to graph $V(x)$. The *Calc* command will provide the approximate point where this function attains its maximum: 0.91287202. Then we calculate the corresponding height and find out that the side of the base and the height practically coincide. In other words, the cube with side of length 0.9128 has the maximum volume among all closed boxes with L = 5. Evidently, if the answer is a cube then the side of this cube has to be $\sqrt{5/6}$. A further step leads us to the construction of a simple four-step program to test our conjecture:

Program MaxVol (TI-83)

: Input "Surface?", L

: $\text{fMax}(0.25x * (L - 2x^2), x, 0, \sqrt{L/2}) \to R$

: $(L - 2R^2)/(4R) \to S$

: Disp R, S

(We chose $\sqrt{L/2}$ as the right bound inside the *fMax* command since L - $2x^2 > 0$ implies $x < \sqrt{L/2}$; any number bigger than $\sqrt{L/2}$ will work as well).

For instance, if L = 10 we will get on the screen the two numbers 1.290993139 and 1.290997068. These numbers are pretty close to each other, as predicted by our conjecture. By experimenting with many different values of L we may conclude that it is highly likely that the height of the box of maximum volume should be equal to the length of the side of its base. In other words, a cube of side $\sqrt{L/6}$ seems to be the answer.

Is it possible to find an acceptable mathematical proof that does not use calculus? The answer is yes, provided that we apply the inequality between the arithmetic and geometric means for any three positive numbers a, b, c (Beckenbach and Bellman, 1961), namely

$$\sqrt[3]{abc} \le \frac{1}{3}(a + b + c),$$

where equality holds if and only if $a = b = c$. We note that

$$V^2 = \frac{1}{16}x^2(L-2x^2)(L-2x^2) = x^2(\frac{L}{4}-\frac{x^2}{2})(\frac{L}{4}-\frac{x^2}{2}) \le [\frac{1}{3}(x^2+\frac{L}{4}-\frac{x^2}{2}+\frac{L}{4}-\frac{x^2}{2})]^3 = [\frac{1}{3}\frac{L}{2}]^3$$

Thus V^2 will attain its maximum when $x^2 = \frac{L}{4}-\frac{x^2}{2}$, i.e. when $x = \sqrt{L/6}$. Of course, this is the same point where V will attain its maximum. If $x = \sqrt{L/6}$ we will have $y = (L-2(L/6))/4\sqrt{L/6} = \sqrt{L/6}$. In other words, the answer is a cube! Undoubtedly, the use of calculus would lead to the solution immediately because $V(x)$ happens to be a simple polynomial of third degree. Indeed, students will appreciate better the power of calculus if they see first a purely algebraic approach to the problem.

Content Area of Math 4417

Part I

1. Linear equations in one and two variables. Assorted real-life problems that lead to linear equations. Basic linear programming.
2. Non-linear equations involving exponentials and logarithms. Annuities.
3. Quadratic equations. Quadratic formula. Justification by "completion of squares". Equations that can be reduced to quadratics by a suitable transformation.
4. Diverse applications to physics (a boat going upstream and downstream, depth of a well and speed of sound, and the like)
5. Complex numbers. Use of the complex field to solve problems in the real domain, especially in the areas of number theory and geometry. Roots of cubics and quartics (comparison between Ferrari's and Descartes' method). Examples from geometry and physics that lead to polynomial equations of third or fourth degree.

Part II

1. Cartesian approach to parabolas and other conics.
2. Maxima and minima of quadratic and biquadratic functions. Solution of optimization problems in plane and solid geometry.
3. Equation of the tangent to a parabola, ellipse, and hyperbola. Parabolic telescopes.
4. Basic kinematics and parabolas. Range of a projectile. Lambert W function and closed solution of equations of the type $(a+bx)e^{cx} = d$
5. Area of a segment of a parabola through a limit process.
6. Comparison between Euclidean and Cartesian methods in geometry.

Part III

1. Geometry of the circle.
2. Justification of the fact that for any circles with perimeters p, p' and radii r, r' the equality $p/r = p'/r'$ is true. Definition of π. Radian measure of angles.
3. Approximation of π through Archimedes method and by simulation. How to avoid the phenomenon of "cancellation".
4. Area of a circle through a limit process.
5. Proof of an important equality: $p/2r\pi = \alpha/360^o$. The conical hat problem.
6. Basic plane trigonometry. Ptolemy's theorem. Sine and cosine of the sum and difference of two angles (different proofs). Law of sines and cosines. Proof of Heron's formula using the law of cosines (comparison with a purely algebraic approach). Application problems (building a tunnel, two-islands, antenna, et cetera). Huygens and the phenomenon of refraction of light.
7. Volume of a pyramid, cone, and sphere. Alternative use of Cavalieri's principle.

Part IV

1. Arithmetic-Geometric-Mean inequality (AGM), with particular emphasis on the cases $n = 2$ and $= 3$. Cauchy's proof.
2. Use of AGM and Heron's formula to solve the isoperimetric problem for any triangle.
3. Finding the minimum of the function $x \to Ax + \frac{B}{x}$ through AGM.
4. Diverse problems that lead to the need to maximize or minimize functions: tents, the rectangular box with maximum volume (with and without a square base), the cylinder with minimum surface area, hanging lamps, et cetera.

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