

Teaching linear equations using *mathematica* at senior secondary school level.

Luckson M. Kaino, BSc(Ed), MSc, PhD

Senior Lecturer and Head, Department of Mathematics and Science Education

Faculty of Education, University of Botswana, BOTSWANA.

kainol@mopipi.ub.bw or kaino_dr@hotmail.com

Abstract

Educational studies indicate that the use of some computer programs can be effective and versatile in instruction. These studies have shown that programs can contribute to innovative student-centered learning environment and provide motivation, generate enthusiasm, interest and involvement and enhance thinking and problem-solving skills as well. In this paper, *mathematica* program is used to investigate systems of linear equations of two variables at senior secondary school level (forms 4 and 5). The reference is made to Botswana senior secondary school syllabus where one of the aims of teaching mathematics puts emphasis on modeling and describing practical life problems. The illustrations described involve singular, non-singular as well as ill-conditioned systems with an application approach to explain the significance of this program in mathematics learning. At the end, the importance of using ICT in instruction is emphasized and some limitations in visualization using this program are outlined.

Introduction

Studies show that the use of computer programs in instruction can contribute to innovative student-centered learning environment (Smeets and Mooij, 2001) and provide power for students to control over their own learning (Broekman et al 2002). It is argued that computer programs and other Information and Communication Technologies (ICTs) provide motivation and variety, generated enthusiasm, interest and involvement, maintained attention and enjoyment, enhanced thinking and problem-solving skills in his study (Sibiya, 2003). Some educationists consider ICT as the only way to go if not a substitute for conventional teaching and learning resources (Broekman et al, 2002). ICTs' interactive testing and review mechanism, together with "a let-s-go-back and look-at-that-again-loop" was believed to offer the best way of learning (The Star Newspaper, 2000).

Some generic programs i.e. general purpose programs such as word processor, spread sheets and graphics have been found to alleviate some students' misconceptions of mathematical and scientific ideas (Ghazali and Ismail, 1997 and Fuglestead, 1997). Subject specific programs i.e. programs designed specifically for a subject area such as *derive*, *sketchpad*, *matlab* and *others* have also been found to be significant in rapid delivery of knowledge than traditional ways involving "chalk and board" or "paper and pencil" in mathematics instruction (Kaino, 1994 & 1998). A number of programs targeting particular mathematical area contents have been developed and advanced. These programs are well noted in areas such as algebra and geometry. For example, the Computer Algebra Systems (CAS) was found to easily generate graphical representations of problems, manipulate algebraic as well as compute numerical representations (Murphy, 1999). Also TI-Interactive is software to teach and learn mathematics with characteristics such as the automatic updating of documents and the existence of sliders allowing to change the values of parameters, making TI-Interactive an essential tool to illustrate different situations that eventually occur in a given problem (Gossez, 2004). Mathcad also is a program that allows individuals to make numerical and symbolic calculations and graph expressions in 2D or 3D. It can be used to communicate technical ideas in a unified, easy-to-use and visual format (Grode, 2004). Combined in a seamless integration with Smartsketch program, Mathcad also

performs computer-aided precision drawing and diagramming. On its own, Smartsketch works both as an interactive dynamic geometry application and as a computer aided design application.

Another widely used software is a Dynamic Geometry software applied in teaching and learning of geometry. The software's facility for easy visualization and powerful manipulation of diagrams has extended its application to other areas of the mathematics curriculum, for example algebra, calculus, and trigonometry (Jackiw, 2004). One of the dynamic geometry software widely used is The Geometer's Sketchpad. With *Sketchpad*, students can construct an object and then explore its mathematical properties by dragging the object with the mouse. All mathematical relationships are preserved, allowing students to examine an entire set of similar cases in a matter of seconds, leading them to generalizations. *Sketchpad* encourages a process of discovery in which students first visualize and analyze a problem, and then make conjectures before attempting a proof (Key Curriculum Press, 2004).

The above outline of computer programs underlines the importance of such technologies might have in mathematics instruction. In this paper, *mathematica* is illustrated to investigate systems of linear equations of two variables at senior secondary school level. Singular, non-singular systems as well as ill-conditioned systems are considered.

Activity

Problems involving systems of linear equations exist in many practical situations. The simpler ones occur involving only a few unknowns and involve little computations. Such problems can be solved by known procedures such as elimination methods or by hand calculators. For large systems, i.e. those with many variables, some powerful calculators or computers might be used to get solutions easily. In this paper we consider mathematical problems involving linear equations with two variables solved in a 'traditional' way and then illustrate on how *mathematica* can be utilized for further investigations.

A linear problem with two variables x_1 and x_2 can be represented by the following system:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2, \text{ where } a_{11}, a_{12}, a_{21} \text{ and } a_{22} \text{ are coefficients of the system.}$$

The above equation can represent different problems in diet, manufacturing, transportation and many others. In this paper, let us consider a diet problem in which it is required to determine the unknown variables x_1 and x_2 which denote two different types of foods. Let the variables b_1 and b_2 represent two basic nutritional ingredient units which have to be achieved by a person for a balanced diet per day. Assuming that each unit of two types of foods contain a_{11} , a_{12} , a_{21} and a_{22} units in both nutrients respectively, then the diet problem can be represented by a system of two linear equations described above. Let us provide coefficient values of this equation and have the system as follows:

$$1.00x_1 + 1.00x_2 = 2.00$$

$$1.00x_1 + 1.01x_2 = 2.01 \dots \dots \dots (1)$$

Equation (1) above has an exact solution $(x_1, x_2) = (1.00, 1.00)$ and in this case we know that no errors are encountered in the solution process. If we change coefficient a_{22} from 1.01 to 1.02, the system changes to

$$1.00x_1 + 1.00x_2 = 2.00$$

$$1.00x_1 + 1.02x_2 = 2.01 \dots \dots \dots (2)$$

Equation (2) also has an exact solution $(x_1, x_2) = (1.5, 0.5)$ and shows that changing the coefficient by the value of 0.01 affects the variable x_1 to be 1.5 times the original value as well as the variable x_2 by making it half the original value. Thus a change of 0.01 in the coefficient, which could be seen as a small value, may contribute a significant change in some problems. Assuming an error of 0.01 in the coefficient a_{22} of the diet problem was incurred, it would mean making the type of food x_1 to be 1.5 times more while reducing the type of food x_2 by half. Of interest, while total ingredient units b_1 and b_2 are maintained (as their values are not affected by the changes in both x_1 and x_2), the solution could bring harmful results to human health because the diet would not be balanced. Such a case can be related to a person who takes a certain fixed quantity of food daily without realizing that it did not contain appropriate nutrient components. Consider $a_{22} = 1.01$ was changed to 0.99 and $b_2 = 2.01$ to 2.02:

$$\begin{aligned} 1.00 x_1 + 1.00 x_2 &= 2.00 \\ 1.00 x_1 + 0.99 x_2 &= 2.02 \dots \dots \dots (3) \end{aligned}$$

Equation (3) has an exact solution $(x_1, x_2) = (4, -2)$. Compared to the former solution $(x_1, x_2) = (1.00, 1.00)$, the latter gives a big difference. If equation (3) was to represent a diet problem and assuming that the coefficients remained the same, then such a solution would be unrealistic (though correct) if assumptions $x_1 > 0$ and $x_2 > 0$ are considered. Consideration of introducing such constraints is important because realistically a minus sign in solutions of x_1 and x_2 would mean to take away that food from someone, thus contradicting the purpose.

Consider $a_{21} = 1.00$ was changed to 1.01:

$$\begin{aligned} 1.00 x_1 + 1.00 x_2 &= 2.00 \\ 1.01 x_1 + 1.01 x_2 &= 2.01 \dots \dots \dots (4) \end{aligned}$$

Equation (4) above has no solution. Such systems with no solution need investigation of equation coefficients which is easily done by calculating the value of determinants. In this case, the determinant of the equation coefficient is zero:

$$\begin{vmatrix} 1.00 & 1.00 \\ 1.01 & 1.01 \end{vmatrix} = 0$$

The above matrix has zero determinant value and the system is said to be *singular*. A *non-singular* system has a non-zero determinant value and solutions for such systems exist. Usually systems such as equation (4) are called inconsistent systems because their determinants are zero or approach zero value.

The above examples reflect the nature of sensitivity the diet problem might possess. In such systems where small changes in coefficients amounts to big different changes from exact solutions are said to be ill-conditioned systems. To model useful diet problems, ill-conditioned systems should be avoided.

Examples in equations (1)-(4) involve coefficients with few decimal places and the nature of the systems might be easily predicted. The situation becomes complicated when coefficients of many decimal places are involved. Let us consider the following system:

$$\begin{aligned} 0.780 x_1 + 0.563 x_2 &= 0.217 \\ 0.913 x_1 + 0.659 x_2 &= 0.254 \dots\dots\dots(5) \end{aligned}$$

Equation (5) has an exact solution $(x_1, x_2) = (1, -1)$. If the solution is changed by 0.001, i.e. $(x_1, x_2) = (0.999, -1.001)$ the difference on the right side of the equation becomes 0.001343 and 0.001572 for the first and second equations respectively. If the output were to be approximated at two decimal places, then there would be no differences in the values of x_1 and x_2 . But if the output were estimated at large decimal numbers, then the differences would be significant. For example, if equation (5) were of the multiple 10^4 the differences would be 13.43 and 15.72 units for the two equations. If the measures are in grams, then the diet problem would be ill-conditioned. Applying *mathematica* program, equation (5) can easily be identified as ill-conditioned as shown in fig. 1 below; they are represented by the same graph. The two equations in (5) show that they are almost the same (graphically, though not symbolically) as they coincide on the same line when plotted on same axes as indicated. Visualization of the difference between two ill-conditioned equations could be a limitation using *mathematica* if the nature of the system is not well recognized and axes extended.

For systems of linear equations with three variables, the concept of three dimensional planes should be brought into light where we expect the intersection of three planes to obtain the solution of three unknown variables. Unlike the solution with two variables which is in 2-dimensions, the intersection of three planes (to determine the solution) as shown in Fig.2 below is not easy to visualize by *mathematica*.

Fig. 1: Graph of two equations in (5) by *mathematica*

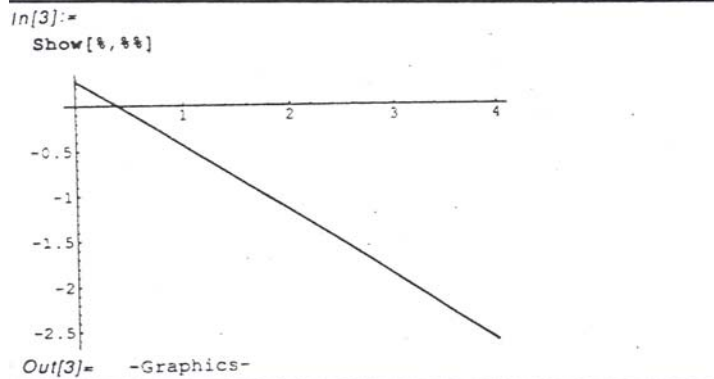
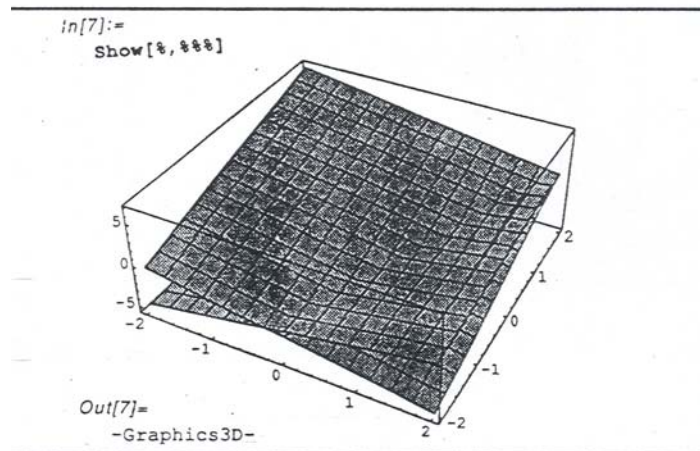


Fig.2: Sketch of three lines intersection by *mathematica*



Summary

Problems involving systems of linear equations can be solved using different methods such as elimination or crammer's rule and others. Hand calculators may also be used to obtain the solution faster than paper and pencil procedures. However, the use of programs such as *mathematica* provides a variety of ways to obtain the solution in numerical, symbolic and in graphical forms at the same time and in a more rapid manner. With *mathematica*, the learner is able to get the opportunity to investigate various stages of the solution process, apply various concepts and acquire skills to solve problems in a variety of ways. As shown in given examples, by changing values of coefficients in the equation matrix, it was possible to investigate the nature of the system dealt with and try to avoid ill-conditioned systems in practical problems. By use of *mathematica*, singular and non-singular matrices of the equation system can be analyzed rapidly in any modeled problem than the use of 'traditional' ways involving "paper and pencil". The Botswana General Certificate of Secondary Education (BGCSE) mathematics syllabus emphasizes the importance of relating each topic taught to real practical and life situations so that students may be able to discover, acquire knowledge and develop skills for application to new situations. And it is plausible that many practical problems involve linear systems that students can easily model for investigation and application using such technology.

Conclusion

Teaching of mathematics using programs such as *mathematica* should target particular topics in BGCSE syllabus and take advantage of current advances in technology. An innovated curriculum in teaching linear equations at this level of schooling should aim at acquiring the following objectives by *mathematica* involving abilities to (i) pose practical problems that involve linear systems, (ii) formulate simple problems by identifying singular and non-singular matrices of the system, (iii) identify simple ill-conditioned systems and errors involved, (iv) use different methods to interpret, analyze and evaluate results obtained, and (v) communicate in both written and oral forms the results obtained.

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