Linking procedural and conceptual mathematical knowledge in technology-based learning

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If we agree that the main goal of mathematics education is to develop both procedural (***P***) and conceptual (***C***) knowledge and to make links between the two, a very important research question regarding technology-based mathematics education is how different technologies affect the relation between the two knowledge types. Our theoretical analysis and practical experience evidence that ***P-C*** links can be established when the learner has opportunities to simultaneously activate conceptual and procedural features of the topic at hand. Such activation is considered for interactive learning that utilizes two technological tools: the ClassPad calculator produced by Casio and Java-based hypermedia lessons developed by (future) mathematics teachers. The contribution describes this kind of learning and examines its empirical values in cognitive and affective terms.

Logical basis for P-C relation

Having made a throrough analysis concerning the studies on conceptual and procedural mathematical knowledge, we (Haapasalo & Kadijevich 2000) made our own characterization, to fit modern theories of teaching and learning:

• *Procedural knowledge* (abbreaviated to ***P***) denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation forms. This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.

• *Conceptual knowledge* (abbreaviated to ***C*** through this paper) denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

It is especially the dynamic and semantic view of ***C***, which we wanted to highlight more clearly. In our view, the two knowledge types can, in some cases, be distinguished only by the level of consciousness of the applied actions. ***P*** often calls for automated and unconscious steps, whereas ***C*** typically requires conscious thinking. However, ***P*** may also be demonstrated in a reflective mode of thinking when, for example, the student skillfully combines two rules without nowing why they work.

Concerning links between ***P*** and ***C*** (the ***P-C*** links) we found four relations:

- *Inactivation view* (*I*): ***P*** and ***C*** are not related (Nesher 1986; Resnick & Omanson 1987).
- *Simultaneous activation* view (*SA*): ***P*** is a necessary and sufficient condition for ***C*** (Hiebert 1986, Byrnes & Wasik 1991; Haapasalo (1997).
- *Dynamic Interaction* view (*DI*): ***C*** is a necessary but not sufficient condition for ***P*** (Byrnes & Wasik 1991).
- *Genetic view* (*G*): ***P*** is a necessary but not sufficient condition for ***C*** (Kline 1980, Kitcher 1983, Vergnaud 1990, Gray & Tall 1993, Sfard 1994).

Having in mind different student abilities, various teaching approaches and topics with associated problems it is appropriate to stress that these four views do not evidence any general conclusion regarding the relation between ***P*** and ***C***. In this paper I highlight some pedagogical implications of the *DI* and *SA* views.

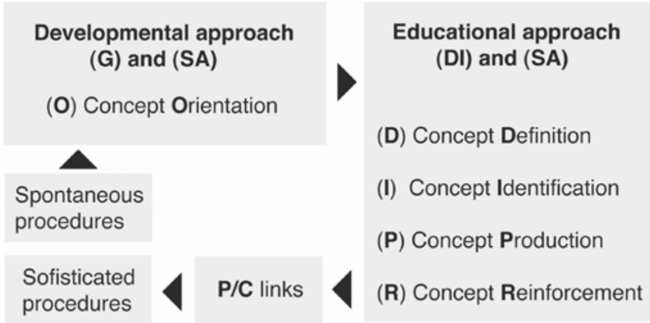
Dynamic interaction and simultaneous activation

Because of the dominance of ***P*** over ***C*** in the development of scientific and individual knowledge, a reasonable pedagogical idea could be to go for spontaneous ***P***, hoping that an appropriate ***C*** would be attainable, finally. On the other hand, it seems appropriate to claim that the goal of any education should be to invest on ***C*** from the first beginning. The *SA* method combines both of these demands in a natural way. However, it is the pedagogical framework that matters when planning how to promote ***P-C*** links in a learning environment. We (Haapasalo & Kadijevich 2000 pp. 147-153) define two pedagogical approaches:

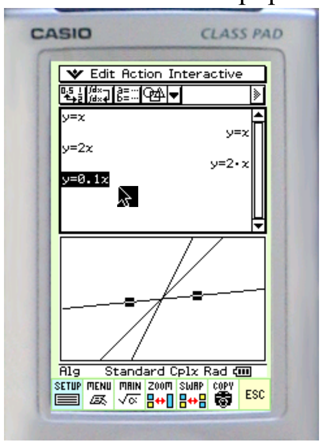
- *Educational approach* is based on the assumtion that ***P*** depends on ***C***. Thus, the logical background is *DI* or *SA*. The term refers to educational needs, typically requiring a large body of knowledge to be transferred and understood.

Developmental approach assumes that ***P*** enables ***C*** development. The logical background is *G* or *SA*, and the term reflects the philogenetic and ontogenetic nature of mathematical knowledge.

The interplay of these approaches can be illustrated in a florishing way only if the framework theory of knowing and learning is linked to the considerations, as the constructivist MODEM framework of on the right, for example. A detailed introduction to the involved *DI* and *SA* methods can be found in Haapasalo (1997, 2003) or Haapasalo & Kadijevich (2003). Furthermore, a thorough learning program for the conceptual field *Proportionality - Linear Dependence - Gradient of a Straight Line through Origin* (denoted shortly by $\boldsymbol{C_1}$) can be downloaded on the Internet at http://www.joensuu.fi/lenni/programs.html. In this paper I will just give some ideas how to utilize progressive educational tools for *DI* and *SA*. I therefore ask the reader to accept a short verbal discription of how the educational approach can be the leading framework, and how the developmental approach is used to trigger the learning process.



Having in mind the remark above, I would like to start with a spontaneous ***P*** and restrict the construction space by simplifying $\boldsymbol{C_1}$: gradient is considered as a concrete slope, at first. Pupils can handle the situation by using spontaneous ***P*** based on their experiences without any explicit thinking of the mathematical relations between the objects. This kind of *orientation* (the first phase of the *concept building*) basically utilizes developmental approach: the interpretations are based on pupils' mental models and more or less naive procedural ideas. These act like a wake-up voltage in an electric circuit that triggers another, more powerful current to be amplified again. ***P*** and ***C*** start to accelerate each other, offering a nice opportunity to use *SA*, for example. Technology allows students opportunities to manipulate the concrete slope visually and look how its abstract symbolic representation is changing. The mental constructions by the student do not need to begin from the concrete or abstract, but between abstract and concrete, and even between abstract things. The figure on the right represents how to utilize *ClassPad* for this purpose by using simple drag-and-drop operations.



In the above mentioned references, examples can be found how to move from the concrete slope to the abstract mathematical concept *gradient* by utilizing the *SA* method again, and how *DI* method is involved in the other phases of *concept building* (*Definition, Identification, Production and Reinforcement*).

Utilizing SA method with ClassPad

For about 20 years, it has been possible to interpret symbolic representations as graphs by using computers. Paradoxically, students should learn to understand these symbolic representations first before being able to utilize computers in this conventional way. This strongly contradicts modern constructivist theories on learning. I would like to illustrate *SA* activities by utilizing *ClassPad 300*, a modern pocket computer made by Casio (see http://www.classpad.org/Classpad/Casio_Classpad_300.htm).

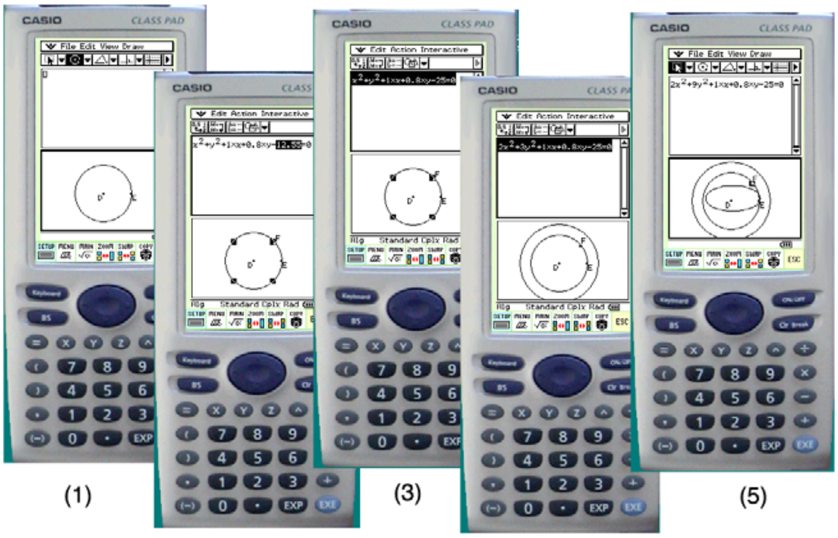
Most *ClassPad* applications support simultaneous display of two windows, allowing to access the windows of other applications from the main application and to perform drag and drop activities (i.e. copy and paste actions), and other operations with expressions between the *Main Application* work area and the currently displayed screen (*Graph Editor, Graph, Conic Editor, Table, Sequence Editor, Geometry, 3D Graph Editor 3D Graph, Statistics, List Editor,* and *Numeric Solver*).

Let's start with an example, which shows how the properties of dynamical geometry programs have been extended to allow an interplay between algebra and geometry.

Example 1. Without knowing anything about the analytic expression of a circle, we can just play harmlessly by drawing a circle in the geometry window (1), and then drag and drop the cirlcle into the algebraic window (2). Something surprising happens: The circle seems to be expressed in algrebraic form $x^2+y^2+0.8xy-12.55=0$. Let's manipulate (3) the equation by changing the constant to 25, then drag-and-drop it to see the new circle (4). It seems that only the radius changes. Let's go back to the algebraic window to do more manipulations (5). This time, let's change the coefficients of the second degree variables: 1 to 2 and 1 to 9: The equation

$2x^2+9y^2+0.8xy-12.55=0$

seems to make an ellipse.

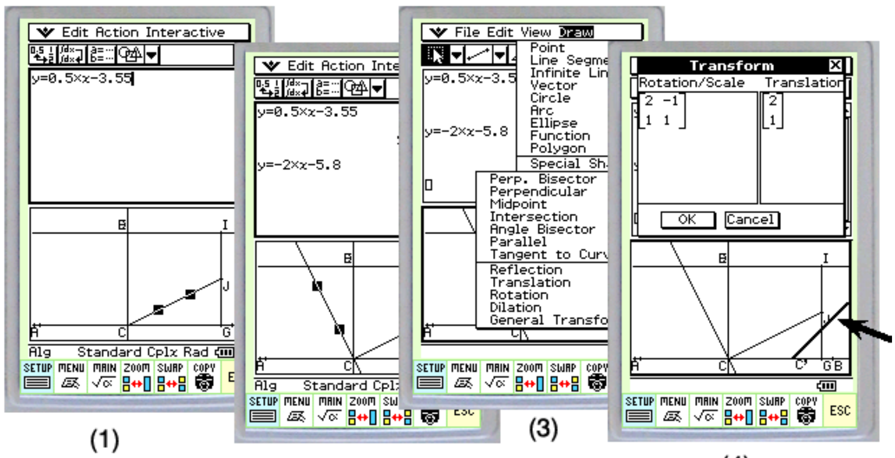


(1) (2) (3) (4) (5)

Anticipating that some readers might question this kind of *informal mathematics*, I would like to point out that the aim of the used *SA* method here has been to enhance *mental links made by the student* and not to produce any exact mathematics, yet. Of course, *ClassPad* modules would allow us to continue the above analysis on a more exact level by using plotting options as 'Sketch' or 'Conics'. The table below shows other types of expressions you can drag and drop between the 'Main Application' and the 'Geometry' window.

Main Application window:	*Geometry window:*
Linear equation in x and y	An infinite line
Equation of circle in x and y	A circle
2-dimensional vector	A point or vector
2 • 2 matrix	A transformation
Equation y = f(x)	A curve
n • 2 matrix	A polygon (each column represents a vertex)

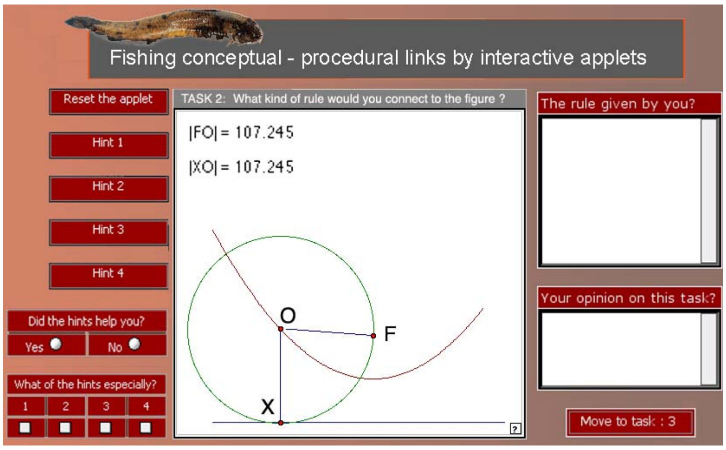
Example 2. Let us construct in the Geometry window (1) the segment CJ. A drag-and-drop activity produces its algebraic presentation 0.5x-3.55. Now we construct a line through C perpendicular to CJ, and are curious to see its equation (2). Interestingly the gradient changed from 1/2 to –2. This gives us a hypothesis, which might be worth of testing. However, this time we would like to play with 'General Transformation' (3). Two matrices appeared in the algebraic window. When filling and dragging-dropping them, the segment moved to a new place (marked by arrow). We make a hypothesis: "A transformation seems to consist of rotation and translation, both being representable by a matrix".



(1) (2) (3) (4)

Utilizing SA method with Java-based hypermedia

Within a joint hypermedia project with the Megatrend University of Applied Science (see Kadijevich & Haapasalo 2003), I planned with my students in the University of Joensuu sophisticated Java applets for teaching and research purposes. This work was a part of students' pedagogical studies in matematics teacher education program. The goal is to collect basic experiences concerning the use of digital hypermedia-based materials in progressive mathematics education. Making the set of interactive applets is an effort to utilize the *SA* method, offering for the student opportunities to make mental links between concrete (often procedural) and abstract (often conceptual) objects by simple manipulations. The figure below illustrates a typical example of the applets' conics (circle, parabola, ellipse, hypebola). At the first stage, just a parabola appears on the screen. The student has opportunity to write his/her open-ended conjecture for the mathematical principle (law) involved in graph, and choose hints at different levels for doing that. It is the mental constructions of the student that matter, not any objectivist-behaviorist definitions to be



written. Maybe the most beautiful and optimal end would be that after just playing for a while with an applets, the student could use a dynamical geomery program (e.g. Sketchapd) for producing the same visualization. The "intelligent applets" will save certain pedagogical parameters, which come out from student's answers and from his/her paths when moving around the applets.

The reader is invited to visit the applet sites at http://www.joensuu.fi/lenni/SA/conics.html and to make own investigations and maybe studies, as well. All kind of feedbak to the author is welcome for developing this "global" progressive learning and research material. The system does not collect any personal data, which could hurt the user's safety.

Closing remarks

We cannot make any definitive conclusions about how, even less in which order, students' knowledge develops in each situation and in each topic. Even the most abstract concepts can be based on their spontaneous ideas. This, however, does not predestine any order for the activities, because it is the pedagogical framework that matters. My position is that (because of the conflict between ***C*** and ***P*** ; cf. Haapasalo 2003), *doing* should be cognitively and psychologically meaningful for the student. Building a bridge between geometry and algebra is just one opportunity to utilize *SA*. I believe that new technology could revitalize the making of mathematics even on students' free time. A detailed analyse of *TIMSS* and *PISA* results reveal (Kupari 2003, Törnroos 2003) that it is not necessarily the school teaching that impacts on students mathematical knowledge. This makes educational research interesting - which factors in our education are important for the development of thinking abilities? If we accept the assumption that the main task of education is to promote a skilful 'drive' along knowledge networks so as to scaffold pupils to utilize their rich activities outside school, it seems appropriate to speak about an educational approach in the sense of this paper.

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