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Heron triangles and Heron´s Formula

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Abstract. In this note I collect some facts about Heron triangles and with it connected Heron´s Formula. First I show the Heron´s Formula with its proof. In the second part of this note I give some remarks about Heron triangles and I show that there exist infinitely many pairs of incongruent Heron triangles having the same area and perimeter.

1. Heron´s Formula

Heron´s Formula concerns the area of a triangle. This formula can be used when we know the lengths of the three sides of a triangle. Compare to the traditional formula (A = ½ (base) * (height to that side)) we never need to determine triangle´s altitude. However is not much known about the author of this formula. Heron of Alexandria lived in the great scholarly city of Egypt, Alexandria sometime between 100 B.C and 250 A.D. What is sure, Heron of Alexandria was a brilliant man who gave the world much insight into the mathematical and physical sciences. Heron wrote many works on mathematical and physical subjects. One of Heron´s treatices called *Metrica* contains Heron´s proof of this formula. Heron´s Formula has a lot of practical applications, for example in geodesy. It is not difficult to compute a three-side area of any lot, if we know the lengths of the sides. Heron´s Formula can also be used if we want compute the area with four or more sides – we decompose this area into the triangular fragments.

Theorem 1. *For a triangle having sides of length a, b and c and area A, we have*

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where

$$s = \frac{a+b+c}{2} \tag{2}$$

is the triangle´s semiperimeter.

Proof: I show that Heron´s Formula is equivalent with the well-known area formula for triangle. I use for this proof the Pythagorean Theorem. (Heron proved this theorem another. He made a geometric proof.)

Let ABC be an arbitrary triangle with sides of length a, b, c and height to side c of length h. (Fig.1).

The foot of the height divide the side c into two segments u and v. Moreover divide the height the triangle ABC into two right-angled triangles.

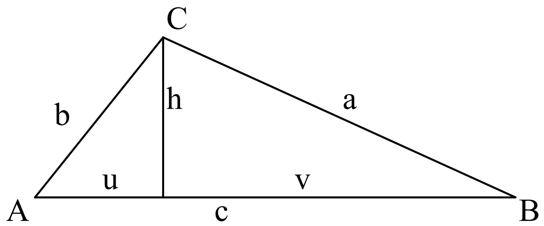

Fig. 1

By using the Pythagorean Theorem and by segment addition, it can be stated that

$a^2 = h^2 + v^2$ $\qquad b^2 = h^2 + u^2$ $\qquad c = u + v$

By the subtraction $a^2 - b^2 = h^2 + v^2 - (h^2 + u^2)$ we obtain $a^2 - b^2 = v^2 - u^2$.

Dividing both sides by $c = u + v$, we have $v - u = \dfrac{a^2 - b^2}{c}$

Adding $u + v = c$ to both sides of equality gives $v = \dfrac{a^2 - b^2 + c^2}{2c}$

Now we use the traditional formula to find the area of a triangle, thus $A = \dfrac{c \cdot h}{2}$.

We take $h = \sqrt{a^2 - v^2}$ and after substituting into the area formula, we get

$$\mathrm{A} = \frac{1}{2}\sqrt{(ac)^2 - \left(\frac{a^2 - b^2 + c^2}{2}\right)^2}$$

Factoring out $\frac{1}{4}$, this gives $$4\mathrm{A} = \sqrt{(2ac)^2 - \left(a^2 - b^2 + c^2\right)^2}$$

Squaring both sides gives $$16\mathrm{A}^2 = \left(2ac - a^2 + b^2 - c^2\right)\left(2ac + a^2 - b^2 + c^2\right)$$

$$16\mathrm{A}^2 = \left[b^2 - (a-c)^2\right]\left[(a+c)^2 - b^2\right]$$

$$16\mathrm{A}^2 = (-a+b+c)(a+b-c)(a-b+c)(a+b+c) \qquad (3)$$

Since the semiperimeter of the above triangle is defined as $s = \dfrac{a+b+c}{2}$, it follows $2s = a + b + c$. We can write the equation (3) as

$$16\mathrm{A}^2 = \left[2(s-a)\right]\left[2(s-c)\right]\left[2(s-b)\right]\cdot 2s \qquad (4)$$

The (4) implies $$\mathrm{A}^2 = s(s-a)(s-b)(s-c)$$

Thus $$\mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

which is the area formula as given by Heron. □

I thing that Heron´s Formula should be teached more at the schools. Most students don´t know this formula. I found out several interesting tasks which are suitable for students in the high school:

1) How can we get out the inside angles of a common triangle when we know the lengths of its three sides?
2) When it´s given an area A of a triangle with the lengths a, b, c of its three sides and an area T of a triangle with the lengths $a + b$, $b + c$, $c + a$ of the three sides, then it obtain $\mathrm{T} \geq 4\mathrm{A}$. Prove it!

2. Heron triangles

Definition 1. *A Heron triangle is a triangle such that the lengths of its three sides as well as its area are integers.*

Proposition 1. *The perimeter of a Heron triangle is even.*

Proof: Assume that the perimeter is odd. Then all factors below radical sign in equation (1) are halves of odd integers. Thus their product is an odd integer divided by 16, which cannot be an integer. □

Proposition 2. *If the sides of a Heron triangle have a common factor t, then his area is divisible by t^2.*

Proof: It is sufficient to show it for t prime. We consider 2 possibilities:
for $t \neq 2$ it is obvious that it obtains,
for $t = 2$ consider the term $s(s-a)(s-b)(s-c)$ modulo 4 and one gets that it can´t be congruent 1.

3. Some remarks on Heron triangles

In this section I show that there are infinitely many pairs of incongruent Heron triangles having the same area. Such pairs of Heron triangles we can obtain by using the Fibonacci sequence. Recall that the Fibonacci sequence (F_n) is defined: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 0$.

Proposition 3. *Let $n \geq 1$ be a positive integer. Then, there exists many pairs of incongrent Heron triangles having the same area $A = F_n \cdot F_{n+1} \cdot F_{n+2} \cdot F_{n+3} \cdot F_{n+4} \cdot F_{n+5}$.*

Proof: Let u and v be two positive integers with $u \geq 2$ and $v \geq 1$. The triangle $\Delta_{(u,v)}$ of sides

$$a = u^2 + v^2 \quad , \quad b = (u\cdot v)^2 + 1 \quad , \quad c = (u\cdot v)^2 + u^2 - v^2 - 1 \qquad (5)$$

has area

$$\mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)} = u\cdot v\cdot(u^2 - 1)(v^2 + 1). \qquad (6)$$

We must prove that on can choose the pairs (u, v) in two different ways such that the corresponding triangles $\Delta_{1(u,v)}$ and $\Delta_{2(u,v)}$ are incongruent but have the same area A. We choose the pairs (u, v) such that

$$(u, v) \in \begin{cases} \{(F_{n+1}, F_{n+4}),(F_{n+2}, F_{n+3})\} & \text{if } n \text{ is even,} \\ \{(F_{n+4}, F_{n+1}), (F_{n+3}, F_{n+2})\} & \text{if } n \text{ is odd.} \end{cases}$$

We check only the case n odd. The arguments for the case n even are similar. We use the well-known formulas $F^2_{n+1} + (-1)^{n+1} = F_n \cdot F_{n+2}$, $n \geq 0$ and

$F^2_{n+2} + (-1)^{n+1} = F_n \cdot F_{n+4}$, $n \geq 0$.

In case for n odd is the area of the triangle $\Delta_{1(u,v)}$ of parameters $(u, v) = (F_{n+4}, F_{n+1})$

$$A = F_{n+4} \cdot F_{n+1}(F^2_{n+4} - 1)(F^2_{n+1} + 1) = F_{n+4} \cdot F_{n+1} \cdot F_{n+3} \cdot F_{n+5} \cdot F_n \cdot F_{n+2}.$$

Area of the triangle $\Delta_{2(u,v)}$ of parameters $(u, v) = (F_{n+3}, F_{n+2})$

$$A = F_{n+3} \cdot F_{n+2} (F^2_{n+3} - 1)(F^2_{n+2} + 1) = F_{n+3} \cdot F_{n+2} \cdot F_{n+1} \cdot F_{n+5} \cdot F_n \cdot F_{n+4}.$$

In order to show that $\Delta_{1(u,v)}$ is incongruent to $\Delta_{2(u,v)}$, it suffices to notice that the shortest side of the triangle given by (5) is a. Hence, it is enough to prove that

$$F^2_{n+1} + F^2_{n+4} \neq F^2_{n+2} + F^2_{n+3} \tag{7}$$

We show that the left side of (7) is larger than the right side of (7). This is equivalent to

$$F^2_{n+4} - F^2_{n+3} > F^2_{n+2} - F^2_{n+1},$$

or

$$(F_{n+4} - F_{n+3})(F_{n+4} + F_{n+3}) > (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}),$$

or

$$F_{n+2} \cdot F_{n+5} > F_n \cdot F_{n+3}.$$

This last inequality is obviously true because $F_{n+2} > F_n$ for all integers $n \geq 0$. □

Proposition 4. *Let $t \geq 1$ be any positive integer and let $\Delta_{1(u,v)}$ and $\Delta_{2(u,v)}$ be triangles of sides*

$$a_1 = t^8 + 5t^6 + 9t^4 + 7t^2 + 2,$$
$$b_1 = t^{10} + 5t^8 + 10t^6 + 10t^4 + 6t^2 + 3 \tag{8}$$
$$c_1 = t^{10} + 6t^8 + 15t^6 + 19t^4 + 11t^2 + 1$$

and

$$a_2 = t^{10} + 6t^8 + 14t^6 + 16t^4 + 9t^2 + 2,$$
$$b_2 = t^6 + 4t^4 + 6t^2 + 3, \tag{9}$$
$$c_2 = t^{10} + 6t^8 + 15t^6 + 18t^4 + 9t^2 + 1.$$

Then, $\Delta_{1(u,v)}$ and $\Delta_{2(u,v)}$ are incongrent Heron triangles having the same area

$$A = t(t^2 + 1)^4(t^2 + 2)(t^4 + 3t^2 + 3), \tag{10}$$

and the same perimeter

$$p = 2t^{10} + 12t^8 + 30t^6 + 38t^4 + 24t^2 + 6. \tag{11}$$

Proof: It is easy to get out that the both triangles of sides given by (8) and (9) have really the area and perimeter given by (10) and (11). In this proof I show how we find them.

According to the Proposition 4 we can obtain for $t = 1$ one pair of incongruent Heron triangles of sides (24, 35, 53) and (48, 14, 50) – both having the same area A = 336 and perimeter p = 112. Based on this example, we find an infinite parametric family of pairs of such triangles.

For a triangle $\Delta_{1(u,v)}$ of sides a_1, b_1, c_1 and semiperimeter s we let $x_1 = s - a_1$, $y_1 = s - - b_1$, $z_1 = s - c_1$, thus we have $a_1 = y_1 + z_1$, $b_1 = x_1 + z_1$, $c_1 = x_1 + y_1$ and $s = x_1 + y_1 + z_1$. Then A = $\sqrt{x_1 y_1 z_1 (x_1 + y_1 + z_1)}$.

For the pair of triangles (24, 35, 53) and (48, 14, 50) we have

$$x_1 = 32,\ y_1 = 21,\ z_1 = 3 \quad \text{and} \quad x_2 = 8,\ y_2 = 42,\ z_2 = 6. \tag{12}$$

Now we can generalize (12) and we get

$$x_1 = \lambda^n,\ y_1 = uv,\ z_1 = u \quad \text{and} \quad x_2 = \lambda^m,\ y_2 = \lambda^k uv,\ z_2 = \lambda^k u, \tag{13}$$

where u, v, λ, m, n, k are integer valued parameters. Since the two triangles have the same area and perimeter, it follows that

$$x_1 \cdot y_1 \cdot z_1 = x_2 \cdot y_2 \cdot z_2 \quad \text{and} \quad x_1 + y_1 + z_1 = x_2 + y_2 + z_2 .$$

From the first equation we get $m + 2k = n$.

From the second equation we get

$$\lambda^{n} + uv + u = \lambda^{m} + \lambda^{k}(uv + u)$$

$$\lambda^{n} - \lambda^{m} = (\lambda^{k} - 1)(uv + u)$$

$$\lambda^{m}(\lambda^{n-m} - 1) = (\lambda^{k} - 1)(uv + u). \qquad (14)$$

Since $n - m = 2k$, it follows that equation (14) can be written as

$$\lambda^{m}(\lambda^{k} + 1) = u(v + 1). \qquad (15)$$

Now, we can choose $u = \lambda^{k} + 1$ and $u = \lambda^{m} - 1$ and the equation (15) holds.

The last condition that we need to prove is that the value of the are of the two triangles is always an integer. Hence, the number $x_1 y_1 z_1 (x_1 + y_1 + z_1)$ must be a perfekt square. From this we get by substitution

$$x_1 y_1 z_1 (x_1 + y_1 + z_1) = \lambda^{n} u^2 v(\lambda^{n} + uv + u) = \lambda^{n}(\lambda^{k} + 1)^2(\lambda^{m} - 1)(\lambda^{n} + \lambda^{k+m} + \lambda^{m}) =$$
$$= \lambda^{m+2k}(\lambda^{k} + 1)^2(\lambda^{m} - 1)(\lambda^{m+2k} + \lambda^{k+m} + \lambda^{m}) =$$
$$= \lambda^{2m+2k}(\lambda^{k} + 1)^2(\lambda^{m} - 1)(\lambda^{2k} + \lambda^{k} + 1). \qquad (16)$$

In order for the number by formula (16) to be a perfect square, it suffices to choose λ, k, m such that

$$(\lambda^{m} - 1)(\lambda^{2k} + \lambda^{k} + 1) \qquad (17)$$

is a perfect square. We choose $m = 3k$ and the formula (17) can be written as

$$(\lambda^{3k} - 1)(\lambda^{2k} + \lambda^{k} + 1) = (\lambda^{k} - 1)(\lambda^{2k} + \lambda^{k} + 1)^2. \qquad (18)$$

The number given by formula (18) is a perfect square when $k = 1$ and $\lambda = t^2 + 1$ for some positive integer t. Hence, $m = 3$, $n = 5$, $u = \lambda + 1 = t^2 + 2$ and $v = \lambda^3 - 1 = (t^2 + 1)^3 - 1 = t^6 + 3t^4 + 3t^2$. Working backwards we find that the triangles given by (13) are the ones given by Proposition 4. □

Remark 1. Since the two Heron triangles (24, 35, 53) and (48, 14, 50) have the same area and perimeter, it follows that the two Heron triangles ($24t$, $35t$, $53t$) and ($48t$, $14t$, $50t$) have also the same area and perimeter for any positive interger t.

Remark 2. Proposition 4 denotes an infinite set of pairs of Heron triangles having the same area and perimeter but these are not all of them. For example, the pairs of Heron triangles {(20, 21, 29), (17, 25, 28)} or {(12, 35, 37), (17, 28, 39)} have the same area and perimeter but they are not particular instances of the set given by (8) and (9).

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