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Differential Calculus Maxima and Minima

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RATIONALE BEHIND THE UNIT

A typical approach by the teacher to teaching maxima and minima is to teach the following procedures of determining a minimum or a maximum:

1. Equate the derivative to zero, then find the critical numbers.
2. Find the second derivative.
3. Evaluate the second derivative at the critical numbers.
   - If the second derivative is negative, then the critical number is the x-component of the maximum turning point. Use the equation of the function to determine the corresponding y-component.
   - If the second derivative is positive, then the critical number is the x-component of the minimum turning point. Use the equation of the function to determine the corresponding y-component.

While this may enable the learners to successfully use the procedures to identify the minimum or maximum, the following are not addressed by the approach:

- Learners should see mathematics as a tool to solve real-life problems.
- The usual question in the learners' mind is, maximum/ minimum of what?
- Learners should have a clear understanding of minima and maxima in terms of relevance to practical situations.

WHAT THIS UNIT IS ABOUT

In this unit the learners will be able to:

- Use practical situations to identify maximum or minimum.
- Use practical situations to identify critical numbers.
- Link the practical maximum or minimum situation to the value of the derivative.
- Link the practical maximum or minimum situation to the arithmetic sign of the second derivative.
- Link the practical maximum or minimum to the minimum or maximum as relates to position on the graph.

In this unit you will

- Use fence of a given length to enclose a rectangular vegetable garden.
- Identify the arrangement of fence that will maximise the enclosed region.
- Find the x-value that will give rise to the maximum area.
- Compare the derivatives associated with the rectangular arrangements, with a view to showing that at the maximum the derivative is zero. This is intended to link real-life maximum with zero derivative.
- Evaluate the second derivatives at the x-values of the different rectangular arrangements with a view to show that maximum will occur where the second derivative is negative.
- Draw the graph based on the different arrangements with the view to show maximum in terms of highest value of the function.

Assessment Evidence You will need to:

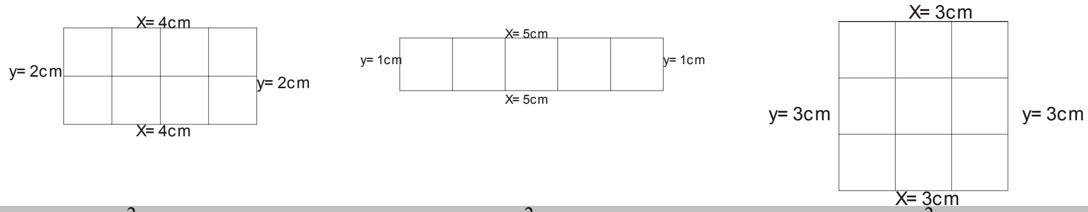
- Correctly identify the arrangement that will maximise the area.
- Correctly identify the arrangement that will minimise the length.

- Correctly identify the first derivative at a maximum /minimum value as being zero.
- Correctly identify the second derivative at a maximum value as being negative.
- Correctly identify the second derivative at a minimum value as being positive.

Activity 1: Determining Area

4.1. You are given a fence of length 12 m to arrange into a rectangular vegetable farm. Use the fence to form as many possible farms of horizontal length x and vertical length y as possible, each time determining the resulting area A, which is the size of the vegetable farm.

Discussion: Some of the expected responses



Area = 8cm^2 Area = 5 cm^2 Area = 9cm^2

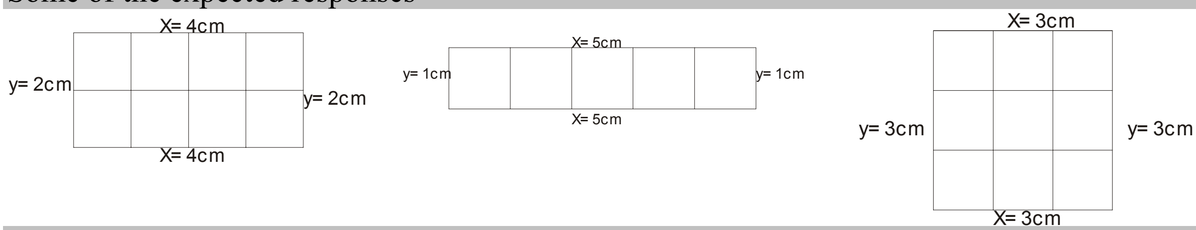
If we use (x:y) to show the rectangular arrangement with horizontal length being x units and vertical length being y-value, then the points associated with the above arrangements are respectively (4;2), (5;1), (3:3). The learners should be able to see that there are many possible arrangements of the fence. For instance, the following are further examples of the resulting points: (2;4), (1;5), (5,5;0,5) etc. In other words, each rectangular arrangement is represented by a point.

Activity 2: Derivatives and Maximum value.

5.1. The Area A is given by A=xy where 2x+2y=12, or $A(x) = 6x - x^2$.

- For each arrangement, determine $\frac{dA}{dx}$.

Some of the expected responses

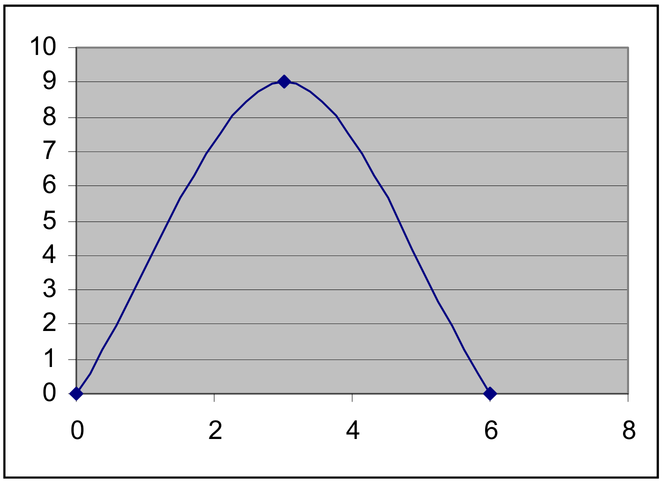


A(4)=8, $\frac{dA}{dx} = -10$ A(5)=5, $\frac{dA}{dx} = -4$ A(3)=9, $\frac{dA}{dx} = 0$

It should be clear to the learners that the only arrangement for which the derivative is zero is the one corresponding to maximum area. The x-value associated with the zero derivative (in this case x=3) is called the critical number. All other will be positive or negative.

- Plot A (x) against x. What kind of a graph results when you join the points?

Discussion $A(x) = -x^2 + 6x$



1. The highest point on the graph is (3,9). It should occur to the learners that this is the point that corresponded with the rectangle of maximum area. Focusing on this connection will enable the learners to see that:

- The maximum in the graph represents an actual maximum area in real life.
- The derivative, i.e. slope of the tangent to the curve, at x=3, is zero. This will confirm what the learners will have determined, using practical arrangements of the fence.
- Learners' attention should be drawn to the arrangements of the fence with a view to observing that all the arrangements with x-values less than 3 have positive slopes, while those x-values greater than 3 have negative slopes. In other words, the learners will observe that as x increases through the critical number 3, the derivative moves from positive to negative. This can be linked to the slopes as observed in the graph. This will facilitate the understanding of the First Derivative Test.

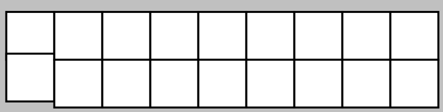
2. The learners should evaluate the second derivative at the critical number 3. They should observe that it will be negative. This will serve as the basis for introducing Second Derivative Test.

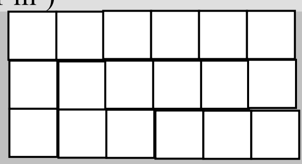
Activity 3: Perimeter.

You are given a vegetable garden adjacent to a river, 18 m^2 in area, to fence. Identify all possible arrangements of the fence forming a boundary for the vegetable garden, if the side of the vegetable garden adjacent to the river does not have to be fenced.

Discussion

Some of the possible responses (One block representing 1 m^2)





Length of fence = 13m Length of fence = 12 m

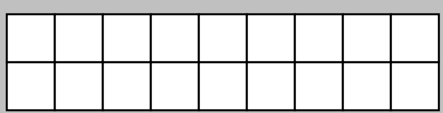
The points associated with the above arrangements are (9;13), (6,12). For (x;y) in this case, x represents the horizontal length while y represents the length of the fence. Examples of other possible points are (2,20), (3;15), (18,20), (1;37), etc. So far the learners should see that x=6m is the one that corresponds to the minimum length of the fence.

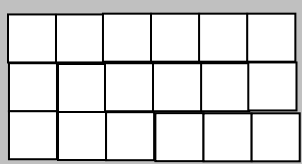
Activity 4: Derivatives and Minimum value.

4.1. If the length of the horizontal part of the fence is x and that of its vertical side y, then the length P of the fence used will be P = x + 2y. The area A enclosed by the fence will be 18 = xy. This means that $P(x) = x + \frac{36}{x}$. For each arrangement, determine $\frac{dP}{dx}$.

Some of the expected responses

Some of the possible responses (One block representing 1 m^2)





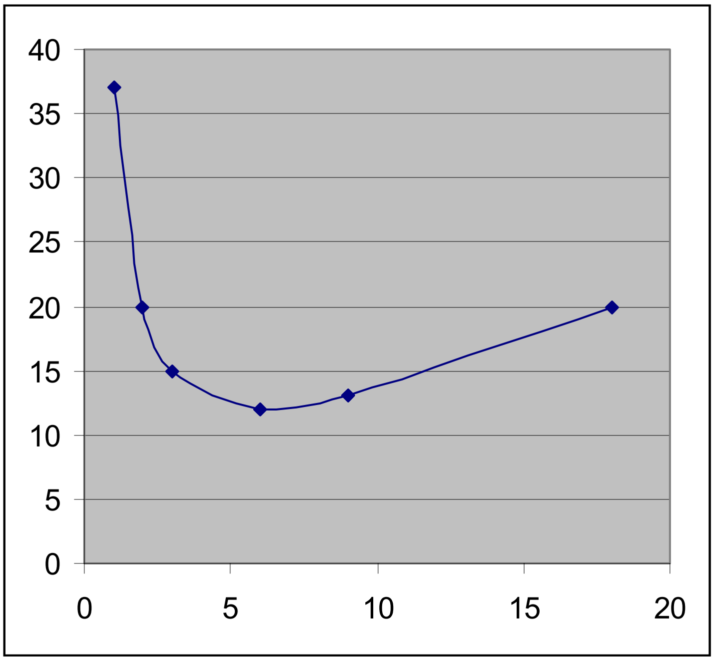
Length of fence = 13m Length of fence = 12 m

$P(9) = 13\text{m} \quad \frac{dP}{dx} = \frac{45}{81}$ $P(6)=12\text{m} \quad \frac{dP}{dx} = 0$

Let the learners determine $\frac{dP}{dx}$ for the other arrangements.

- Plot P (x) against x for all possible values of P (x).

Discussion



The minimum will be at x= 6m. It will be 12m. It will occur where $\frac{dP}{dx} = 0$. All the derivatives on the left of x=6 will be negative, while those on the right will be positive. This again forms the basis for First Derivative Test. Second Derivative Test follows similarly to the earlier discussion on area.