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Mathematical Problems and Mathematical Structures

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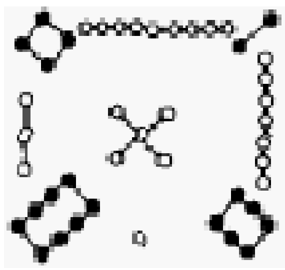
1. Mathematical Problems and Problem-Solving

Mathematics structures have been discovered and developed through solving mathematical problems. One of the main roles of schools is to preserve human culture, including the mathematical culture. Thus, in teaching/learning of each mathematical topic, schools had used to offer mathematical problems. Since the end of the 'New Math' movement, at the end of the 70s of the last century, this situation has changed in different parts of the world, especially in the so-called 'Western World' (Malaty 1998 and Malaty 1999). As a reaction to the 'New Math', the 'Back-to-Basics' movement replaced the 'New Math' to put emphasis on mastering arithmetical skills. As mastering arithmetical skills is not a sufficient objective for mathematics education, towards the year 1985 a new slogan was worked out to make some balance with the mechanical character accompanied the teaching of arithmetical skills. This was the slogan 'Problem-Solving'. According to the new slogan, mathematics educators became interested in collecting and developing problems to be used in teaching mathematics at schools. The type of problems chosen was of puzzle type, where no need to structured mathematics. In addition, this type of problems was described as unrelated to a particular educational level (Pehkonen 1992, 4). These mentioned characteristics were relevant to the unstructured school mathematical curricula, which among others were appreciated to meet with the lack of qualified mathematics teachers in particular Western Countries. On the other hand, these characteristics of problems attracted educators, who were not specialised in mathematics education, to the bright slogan 'Problem-Solving'. Some of these educators have taken part in developing a theoretical frame for 'Problem-Solving'.

2. Structured or Unstructured Mathematics?

The 80s and 90s of the last century saw the spread of problems like matchsticks problems (Malaty 2002) and others, where no need of structured mathematics. The claiming of being unrelated to a particular educational level is another question. One of the problems found relevant for all was the problem of the 3 $\times$ 3 magic square. The magic pattern of the problem attracted teachers to offer the problem to even first graders of primary school. In such cases, where young children were asked to solve the problem at home, the real solvers were children's parents and other adults. In some cases, children announced that nobody was able to solve the problem at home.
The problem could be solved through inquiry of different cases, but it needs to at least simple judgement to appreciate some cases and exclude others. Trial and error is not the proper strategy needed, but systematic inquiry of mathematical thinking. The magic number patterns of the problem was the reason to make Chinese saw in their Lo shu magic square (Figure 1) the origins of science and mathematics (Swetz 1994, 35).

Lo shu means Lo Books and Lo was one of flooding rivers of ancient China. By a legend, the magic square of Figure 1 was a curious figure on the divine turtle shell.



(Figure 1)

Lo shu magic square figure by its form and mysteries represents understanding of number patterns of a structured mathematics. At the corners of the squared shape, black pearls strings represent even numbers, other strings of white pearls represent odd numbers. The form of black pearls represents divisibility by two of even numbers. The odd numbers mystery is related to having all the odd numbers from 1 to 9 on a cross, where 5 is in the middle. Odd numbers were regarded as symbol of completeness and non-divisibility. This was the reason of Arabic talisman related to *'khamsah wa khomisah'*, or 'five and the small 5', which is seen today in Arab World in a decoration of human palm form.

The solving of the magic square problem could be made in different ways, where understanding of number relations, is always needed. One of the appropriate strategies for solving the problem is to find out the number at the central small square.

$(a + e + i) + (g + e + c) +$
$(d + e + f) + (b + e + h) =$
$15 + 15 + 15 + 15$
$\Rightarrow (a + b + c) + (d + e + f)$
$+ (g + h + i) + 3\,e = 60$
$\Rightarrow 15 + 15 + 15 + 3\,e = 60$
$\Rightarrow 3\,e = 15$
$\Rightarrow e = 5$

As 5 is an odd number the sum of a and i must be even and as well the sum of g and c.

a	d	g
b	e	h
c	f	i

(Figure 2)

The sum of two numbers is even if both numbers are even or both numbers are odd. Assume that a and i are odd numbers and as well g and c. In this case the sum of a and g in the top row is then even and therefor d must be odd. In this case at least all the 6 numbers e, a, i, g, c and d are odd numbers. This case is impossible, as from 1 to 9 there are only 5 odd numbers.

Assume that a and i are odd numbers, and g and c are even numbers. The sum of a and g is then odd and therefore d has to be an even number. Similarly f, b and h have to be even. This means that from 1 to 9 we have to find 6 odd numbers, which is impossible. In a similar way g and c could not be odd numbers.

From the above discussion a, i, g and c have to be even numbers. Thus, a has to be one of 4 cases. It could be 2, 4, 6 or 8. Respectively i could be 8, 6, 4 or 2. In each of these cases g could be of two different cases. This means that there are 8 different solutions to the problem. Children can show that these 8 cases are only one case with three images by rotation, where the other four cases are images by reflection to the obtained four cases.

3. To Whom? When? and How?

In writing my paper, I used the Internet sites 'Google search' to verify some parts of the legend around the Lo shu square. Search for 'Lo shu' brought 3 370 sites and search for square "Lo sh" brought 953 sites. At least, most of the sites, related to the search square "Lo shu", are related to mathematics teaching, among them sites of the Math Forum. It is difficult to say that I was able to investigate all the sites, but those to which I get to access did not gave me precise answers to the questions related to the age of students to whom the problem could be offered. As well, there was no answer to the question, when this problem is adequate to be offered. Even the question related to the way of introducing the problem 'How?' was not discussed clearly.

The written proof above is mine, and I got to know about the 'Lo shu' legend, and about the idea of proof, at the age of 14. This was only for students of second grade of senior secondary school, who got more lessons in mathematics. It was a part of a week lesson on the history of mathematics (Lotfy and Abu Alabbas 1958, 37 - 39).

Proposing the problem to us at that age and with our structured knowledge on algebra was a reasonable choice to understand the magic of the Lo shu square. The problem's given allow to construct only 8 first grade equations. The given seems not enough to find values for 9 variables to fit in the 9 small squares. The form of the square allowed us to get the ninth needed equation, i.e. the equation $3e = 15$ used above in the proof.

Indeed the problem could be offered to first graders of primary school, but it has to fit with their structured knowledge. For instance, offering numeral 5 written in the middle square could help few students to find out the solution, but this could be a good relevant problem to students of age 10. For first graders of primary school, the Lo shu magic square could be a good relevant problem for most of the students if we offer beside 5 in the middle square one of the even numbers and one odd number like in Figure 3.

		2
	5	
	1	

(Figure 3)

The question is not absolute in referring to a problem as a good one. We have to answer first the questions 'To Whom?', 'When?' and 'How?'

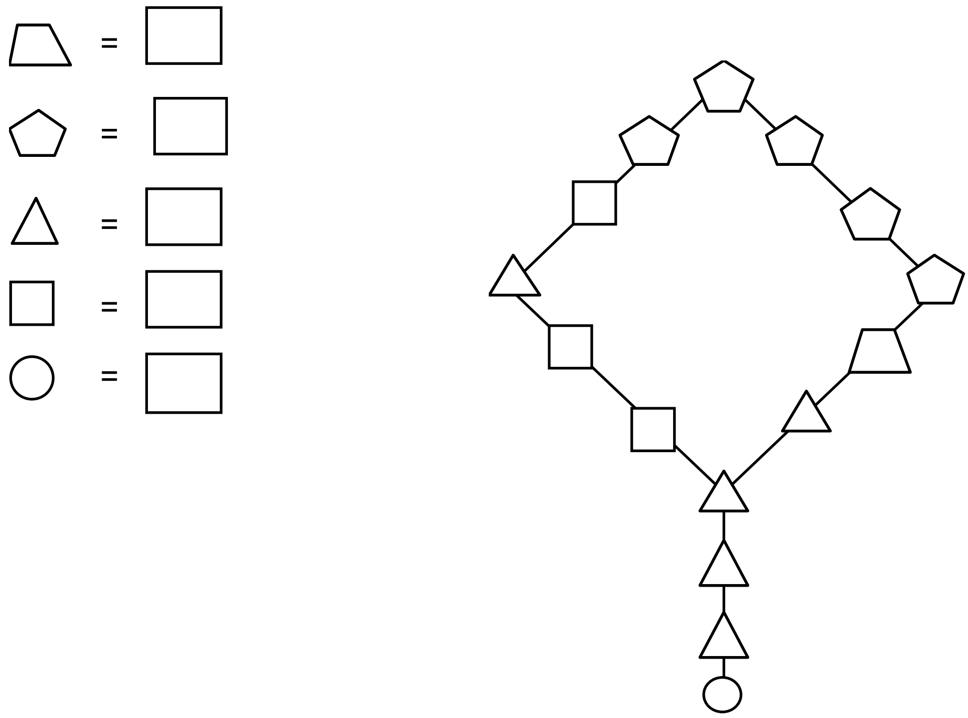
For children of age 11 to 14, depending on the content of the curriculum and the level of children group, we can lead the children to find the mentioned above proof. Yet this is not enough. We have to ask students to find other proof, which could be slightly different one. If this request does not help, we have to go back to the previous proof and ask children to think about the other way we can use to continue the proof after the step of finding out 5 as the value of the number of the middle square. Again if this consoling does not help, we have to ask about the numbers we investigated in the previous proof after finding the number of the middle square. If this again does not help we have to ask about the possibility of choosing other numbers for investigation, other than a, i, g and c (Figure 2).

4.1.Arithmetical Problems

Since the beginning of primary school we can offer appropriate mathematical problems, which is in need for elementary elements of mathematical structures, and which can help in building up mathematical structures. Here is one of such problems offered to first grader students of Hungary (Hajdu 1998 ,66). We have made some changes in the figures used, but the idea is still the same as in the original form. In Hungary, the most appreciated principle on mathematics teaching is the Spiral principle. We can see from the example's discussion below, how teaching of a simple problem for first grader could be a basic experience for solving system of equations in the future. This simple example includes the idea of solving system of 5 First Grade Equations on 5 variables.

In Figure 4 each different shape represents different number larger than 0. The sum of the four numbers on the same line equals 8. What is the number represented by each figure?

It is important in teaching solving such problems to encourage students to find different strategies for solving the problem, discuss all the students proposed



(Figure 4)

strategies, and facilitate the discovery of other ones. In each time, when a student proposes a strategy, the teacher has to ask about the reason of its choosing. Not only right conclusions have to be discussed, it is more worthwhile to discuss wrong conclusions. This is the way to develop all the classroom students' abilities to solve problems and building up mathematical structures. The setting of the figures of the problem shows that solving a problem has not to start from left to right, as the mechanical teaching of arithmetic propose to children.

The best strategy is to start by finding out the number represented by pentagon. The most important to discuss with students is why they prefer to start with this figure. The reason here is that pentagon is the only figure appearing on the upper line to the right. Asking for finding other strategy could lead to find first the number represented by a triangle. The reason this time is the appearing of the triangle 3 times on the vertical line (down). The number, which the triangle represents on this line, must be less than 3. Three is impossible as it makes the value of three figures equals 9. Two could not be the right number as it leads to obtain also 2 as the number represented by a circle. Thus the triangle represents 1. Other solution strategies have also to be discussed. We have mentioned before to the fact that, neither each problem is in need for structured mathematics, nor each problem can facilitate the building of mathematical structures. Here we have to add that not every teaching can develop students' mind. Asking children to justify their strategies and cultivating the habit of searching for the cause is the way we have to take to make mathematics teaching promote children's mind (Malaty 1988).

4.2. Types of Arithmetical Problems

Arithmetical problems, which are in need of understanding structured mathematics and facilitating building up mathematical structures are not only related to equations, variables or figures. Also mentioning to 'Word Problems' as a proper arithmetical problem is not absolutely right. From one hand not each arithmetical problem is a Word Problem, and from the other hand the so-called 'Word Problem' could be not a problem at all.

Let us give an example of a 'Word Problem'. Mary's grandfather gave her some amount of money as a birthday gift. Three fifths of the offered money Mary used to buy a valuable mathematical dictionary. Five sevenths of the remained money she saved on her account and she still have a remainder of 12 Euros. How many Euros the grandfather had offered to Mary?

Can we say that the mentioned above 'Word Problem' is a proper problem? It could be and could be not. If children have not solved before a similar one it could be then a proper problem; otherwise it is a drill on performance skills and not a problem.

Arithmetical problems are not only related to equations, variables, or 'Word Problems'. Let us investigate other type of arithmetical problems by an example. Use the number 2 three times to construct expressions, of the values: a) 6, b) 8, c) 3, d) 1, e) 2.

5. Timing Factor and Problem's Value

The value of the problem and its affect on developing children thinking depends among others on timing. For instance offering the mentioned above 'Word Problem' after learning using letters as a symbol of a variable makes the problem less meaningful in developing children's thinking. Such problem has to be offered earlier and just after studying the relation between multiplication and division, and multiplying by fractions. So, it could be relevant to children of Grade 5. Some of these students may come to the next shortcut: Grandfather's offer = $\frac{5}{2} \times \frac{7}{2} \times 12$ Euros. The most important is to ask these children to explain, to all of the classroom children, how they came to get this statement.

The same problem could be relevant to some students of Grade 3 or even lower depending on student's level. The most needed pre-request in this problem is the understanding of fraction concept. For younger children solving the problem reflects having high abilities of problem solving in structured mathematics.

The other problem mentioned in the previous chapter (4.2.) can be of different meaning regarding timing. This problem is a relevant one to children of Grade 2, who have learnt about the four arithmetical operations. They can give solutions in a form like the next: a) $2 + 2 + 2$, b) $2 \times 2 \times 2$, c) $2 + 2 \div 2$, d) $2 - 2 \div 2$, e) $2 + 2 - 2$.

The same problem could be presented to children of higher grades to achieve other objectives, like measuring the understanding of the role of brackets. In this case, we can beside the mentioned above solutions get others. For part b the solution could be $2 \times (2 + 2)$ and the solution of part e could be $(2 + 2) \div 2$.

Timing could be a decisive factor not only in ranking problem's level, but even in converting an exercise on operation performance into a proper mathematical problem. Let us introduce an example to explain this fact.

Write the missing symbol >, < or = to obtain a true statement: a) $4 + 9 \; \square \; 6 + 9$, b) $6 + 7 \; \square \; 8 + 3$, c) $\frac{3}{5} \; \square \; \frac{4}{9}$, d) $\frac{17}{19} \; \square \; \frac{11}{13}$. Here, parts a and b are proper problems when we offer them before learning addition up to not more than 9. Parts c and d are proper problems, when they are introduced before learning about 'changing into similar fractions'. It is remarkable to notice that problems like those of parts a and b are of special meaning in creating a pre-request for achieving understanding in Algebra, in learning solving equations and inequalities. Parts c and d from one hand are measuring the understanding of the concept of fraction and from the other hand partaking in building the concept of fraction.

6. Textbooks and Teachers: From Simple Drills into Mathematical Problems

Today it is common to find textbooks, which are putting emphasis on mastering arithmetical skills. 'Problem-Solving' in textbooks are mainly unrelated to certain school mathematics topics and therefore they are wrongly thought to be relevant to every child of every school Grade. Strategies of introducing and discussing these problems are not worked out in teachers' books, only answers are provided there as tricks. We do need to have other types of textbooks, which take care of the nature of mathematics and its historical growth as a culture. From the first grade, textbooks have to make children search always for elegant solutions even for exercises, which could seem putting emphasis on algorithmic drills. The last example of the previous chapter could illustrate this idea for lower grades of primary school, here we add other examples for the higher grades of primary school (Malaty 2003). Simplify in elegant way the next expressions: a) $8.91 + 25.7 + 1.09$, b) $7.15 - 9.42 + 12.85 - 0.58$, c) $18.9 - 6.8 - 5.2 - 4.1$, d) $3.17 + 10.2 + 0.83 + 9.8$, e) $15.21 - 3.9 - 4.7 + 6.79$, f) $76.73 + 3.27$, g) 0.16×0.25, h) $0.93 \times 54.7986 + 0.07 \times 54.7986$, i) $24 \times 17 + 17 \times 6$, j) $9 \times 7 + 43 + 34$.
Finding elegant solution here is a joyful work of Problem Solving. The joy getting from solving one part gives motivations to solve other parts. Children have to learn to analyse the problem to find the elegant idea and learn to make syntheses by logical writing of solutions. These abilities are needed in learning mathematics to assist students in building up mathematical structures in their mind. This could attract some students to be those, upon their shoulders could lie the task of deserving and developing mathematics. To simplify these expressions, the type of thinking needed is the same which children shall need later in dealing with more symbolic expressions in learning algebra. In an indirect way children learn to develop, understand and appreciate the properties of number system such as commutativity, associativity and distributivity.
Teachers are the most decisive factor in implementing such teaching. Education of teachers has to help them to understand the nature of mathematics and its structures. As well, this education has to encourage them to act in a creative way. Among others they have not only to encourage student to find different solution to a problem but as well to be able to facilitate the discovering of these solutions. We need also to encourage the creativity of teachers to modify textbooks problems and add new ones of their own. For instance, let us reflect briefly, but as well critically, on the discussion in chapter 5 regarding the second problem of chapter (4.2.). We have not discussed the next solutions: for part a $2 \times 2 + 2$ or $2 + 2 \times 2$, for part b $(2 + 2) \times 2$, for part c $2 \div 2 + 2$, for part e $2 \times 2 \div 2$ or $2\div 2 \times 2$. Also the teacher has to ask children about 'what was the larger number obtained in these expressions' and 'what was the smallest one'. The teacher can ask about the possibility of getting larger number and the possibility of getting smaller number. Here children can find that 0 is the smallest possible number $(2 -2) \times 2$, $2 \times (2 - 2)$, $(2 - 2) \div 2$. They also can learn analogy and propose to investigate the possibility of getting the missing numbers between 0 and 8, which are 4 and 5. This can open the door to children to construct other problems like using 3 instead of 2 or using 2 four times instead of three. This is the type I can appreciate regarding the so-called open-ended problems or in Pólya's terms constructing mushroom of problems.

7. Final remarks

In this paper I discussed mainly arithmetic problems. This is due to two reasons. The first one is that the mechanical way of teaching arithmetical skills, since the 'Back-to-Basics' movement, was the reason of the spread of the puzzles of 'Problem Solving'. Therefore it is important to show that developing arithmetic teaching can offer proper arithmetical problems, which can help in building mathematical structures, among them algebraic structures. The second obvious reason is that in one paper is difficult to discuss more than a limited area.

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