Banach Fixed Point Theorem and the Stability of the Market

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Abstract In the paper the conditions for convergency to market equilibrium are examined applying Banach fixed point principle and relevant iteration process. *Keywords :* Stability, equilibrium, contraction, fixed point, iteration process, demand and supply function, equilibrium point

1.Introduction

Stability analysis is an important part of mathematical economics. Much of economic theory is based on the comparative statics of equilibrium states. The notion of an equilibrium may be defined in various ways. According to Machlup(1963) it is a constellation of selected interrelated variables so adjusted to one another that no inherent tendency to change prevails in the model which they constitute. In essence, an equilibrium is a situation that is characterized by a lack of tendency to change. Such investigations make sense if the underlying situation is stable. By *stability* (Hands(1991)) it is usually understood the issue of whether the values of the variable under consideration converge to its "equilibrium" value over time. Thus, the issue of stability concerns dynamic models, so it is referred to as *dynamic stability* (further only *stability*). The economic argument for convergence (and thus stability) is that competitive market forces will move the price toward its equilibrium value. We will deal with the stability of the simple supply and demand model. In economic literature (e.g. Chiang(1984) or Hands(1991)), these stability-conditions are mostly expressed in terms of properties of demand and supply functions. We will present a stability-argument using the famous Banach fixed point theorem together with its applications in numerical mathematics, namely iteration processes.

A *metric space* is a pair (M,d), where M is a nonempty set and $d : M^2 \to R$ (reals) is a *metric* on M satisfying the following conditions:

(i) $d(x,y) = 0$ iff $x=y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,z) \le d(x,y) + d(y,z)$

for all x, y, z ϵ M. A sequence (x_n) of points of (M,d) is a *Cauchy sequence* if $d(x_m, x_n) \to 0$ for $m, n \to \infty$. A metric space is *complete* if every Cauchy sequence has a limit x ϵ M, i.e. if $d(x_m, x_n) \to 0$ as $m, n \to \infty$, then there exists x ϵ M such that $d(x_n, x) \to 0$ as $n \to \infty$. An operator $A : M \to M$ is a *contraction* if there exists $0 < c < 1$ independent of x, y ϵ M such that for all x, y ϵ M there holds $d(A(x), A(y)) \le c d(x, y)$. An element a ϵ M is a *fixed point* of an operator A, if $A(a) = a$.

2. Iteration principle

Analogous to e.g. Colatz (1966) the following theorem holds :

2.1. Banach fixed point theorem. *Let M = (M, d) be a complete metric space, A: M → M be a contraction. Then there exists a unique fixed point a of A.*

Rephrasing 2.1 and referring to its proof the general iteration method may be formulated which is of great importance in numerical analysis :

2.2. Iteration method. Let (M, d) be a complete metric space, A : M → M be a contraction. Then there exists a unique solution a ϵ M of the equation $A(x) = x$, which is the limit of the sequence (x_n), where $x_n = A(x_{n-1})$, n = 1, 2,…, x_0 is an arbitrary element of M and there holds

$$(2.1)\quad d(x_n, a) \le \frac{c}{1-c}\,(x_{n-1}, x_n)\,;$$

x_n is said to be the *n-th iteration.* The following obvious practical aspects of iteration method will be later recalled :

(i) The expression (2.1) can serve for the estimation of error in the n-th iteration.

(ii) If there is a numerical mistake in the calculation of the i-th iteration x_i and $x_i \in M$, then it has no influence on convergency (as the iteration process may start from $x_0 = x_i$). From this point of view the iteration method is selfcorrecting.

(iii) Sometimes it is convenient to consider an operator A such that A is a contraction not on the entire space but only on some closed neighborhood of an expected fixed point. Then the iteration method can be applied under the condition that A maps this neighborhood into itself and hence the iterations do not fall outside the neighborhood.

Our further considerations will concern metric space (R, d), where R is the set of real numbers and d is the usual metric, $d(x , y) = | x - y |$. It is currently known that (R, d) is a complete metric space. We will consider a closed interval (see (iii) above) $< a, b > = M$ of reals and an operator $f : M \rightarrow M$, where f is a real function for which there exists the derivative f' on M. The following Proposition is valid :

2.3. Proposition. *Let $M = < a, b >$, $f: M \rightarrow M$ and let there exists the derivative f' on M with the property*

$$(2.2) \quad |f'(x)| \leq k < 1$$

for some k. Then f is a contraction.

Remark. The condition (2.2) is usually referred to as *Lipschitz condition*.

As an application of 2.1, 2.2 and Proposition 2.3 the iteration method to find a solution of the equation $x - f(x) = 0$, i.e. $x = f(x)$ can be formulated :

2.4. Iteration method for x - f(x) = 0. Let $M = < a, b >$ and f satisfy conditions of Proposition 2.3. Then there exists a unique solution a of the equation $x = f(x)$ which is the limit of the sequence (x_n), where $x_n = A(x_{n-1})$, n = 1,2,… and $x_0 \in M$. Moreover, a is a fixed point of f. For the estimation of error the following formula holds :

$$(2.3) \quad | x_n - a | \leq \frac{k}{1-k} | x_{n-1} - x_n |$$

3. Equilibrium price as a fixed point in a stable market

We will deal with a simple demand and supply model. The model is given by a *demand function* $P = D(Q)$ and a *supply function* $P = S(Q)$, where P is a *price* and Q is a *quantity (demanded or supplied)*. In the "normal" case demand function is decreasing and supply function is increasing. Their graphs as the *demand* or *supply curves* are depicted in Figure 3.1. The market equilibrium represents the *equilibrium point* $[Q_E , P_E]$, where demand is equal to supply ; Q_E is an *equilibrium quantity*, P_E an *equilibrium price*. Mathematically, the equilibrium point may be found as the intersection point of the demand and supply curves. From this point of view the matter seems to be quite clear. But it neglects the crucial aspect, namely the functioning of the market. The "theoretical" equilibrium point may be simply found using various mathematical tools (finite or iterative methods for systems of equations), but it need not be reached regarding the working of the market.

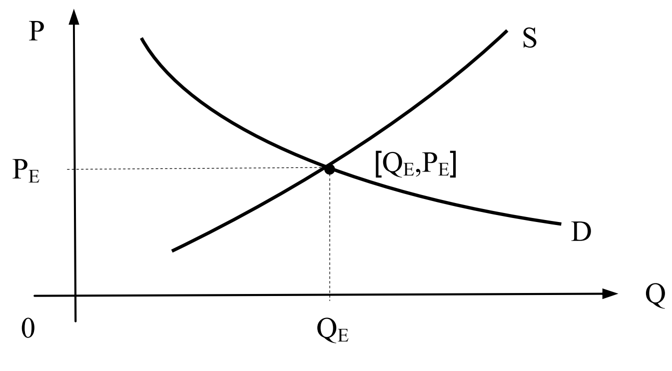

Figure 3.1

Under real conditions, the (market) price differs from the equilibrium price and it is moved to its equilibrium value by the effect of competitive market forces. Consequently, the approaching of the market price to the equilibrium price depends solely on the character of competitive market forces that are given by the demand and supply functions. When the price approaches (converges) to the equilibrium price from every disequilibrium position, the market is said to be *stable*. So, the stability of the market is related to the properties of the demand and supply functions. In the affirmative, the convergency is realized as the sequence of market prices $P_0, P_1, \dots$ converging to P_E. Hereby, the door is open to capture the problem applying Banach fixed point principle and iteration method. For this purpose we will describe the adjustment mechanism that characterizes these competitive forces aiming to find the above mentioned sequence of market prices and to investigate its convergency. The market adjustment mechanism proceeds as follows (see Figure 3.2) :

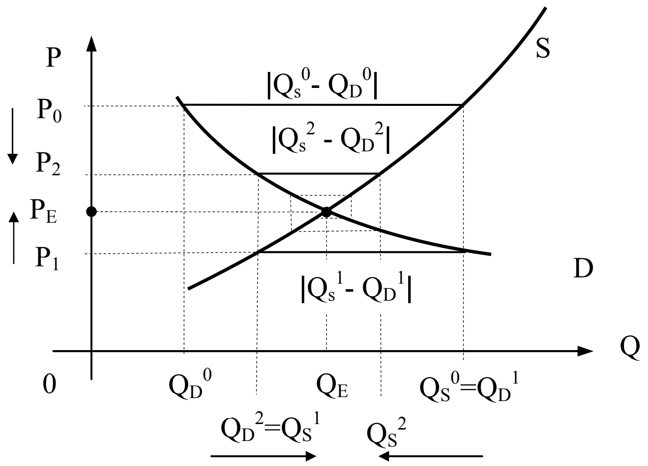

Figure 3.2

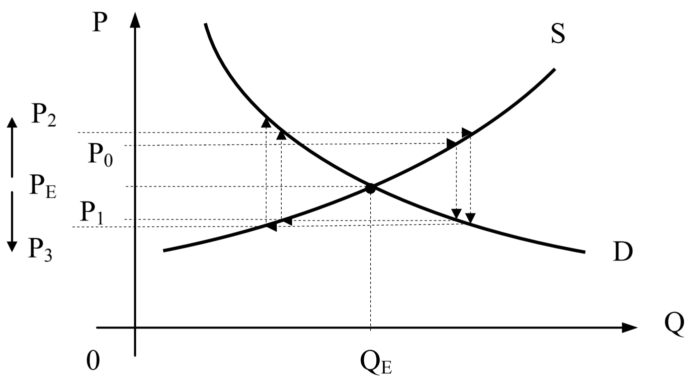

Figure 3.3

Let P_0 be a market price with $P_0 \neq P_E$ and assume (without loss of generality) that $P_0 > P_E$. Denote Q_D^0 a *demanded quantity* and Q_S^0 a *supplied quantity* corresponding to P_0. We have (see Figure 3.2.)

$$(3.1) \qquad P_0 = D(Q_D^0),\ P_0 = S(Q_S^O) \ \Rightarrow\ Q_D^0 = D^{-1}(P_0),\ Q_S^0 = S^{-1}(P_0),$$

where D^{-1} or S^{-1} is the inverse function to D or S respectively (they exist for D,S are strictly monotonic). It holds $Q_S^0 > Q_D^0$, so there is a surplus $|Q_S^0 - Q_D^0|$ of the good on the market. The producers react by the decreasing of the price to the level P_1 , for which the quantity demanded , i.e. $D(P_1)$, equals Q_S^0. In symbols,

$$(3.2) \qquad P_1 = D(Q_S^0) = |\text{ due to (3.1)}| = D(S^{-1}(P_0)).$$

Now, $P_1 < P_E$. For the corresponding demand and supply quantities Q_D^1 , Q_S^1 it holds $Q_D^1 > Q_S^1$, so there is a shortage $|Q_S^1 - Q_D^1|$ of the good on the market. The producers react by the increasing of the price to the level P_2 and there holds (analogous to as given above)

$$P_2 = D(Q_S^1) = D(S^{-1}(P_1)).$$

In this manner we can proceed the process obtaining the sequence P_0 , P_1 , P_2 , … ,P_k ,… of market prices , where

$$(3.3) \qquad P_k = D(S^{-1}(P_{k-1})),\ k = 1,2,\dots .$$

According to Proposition 2.3 if $|(D(S^{-1}))'(P)| = k < 1$, then $D(S^{-1})$ is a contraction. Since

$$(D(S^{-1}))' = \frac{D'}{S'} ,$$

we get

$$(3.4) \qquad \left| \frac{D'}{S'} \right| = k < 1 .$$

Referring to 2.1 and 2.2 the following conclusion may be stated :

3.1. Conclusion. Let P = D(Q) be a demand function and P=S(Q) be a supply function satisfying the property

$$\left| \frac{D'}{S'} \right| = k < 1 .$$

Then the sequence (P_k) given by $P_k = D(S^{-1}(P_{k-1}))$, k = 1,2,… of the market prices converges to the equilibrium price P_E , where P_E is a fixed point of the operator $D(S^{-1})$ and the market is stable.

4. Interpretation, application

4.1. The property (3.4) may serve as the stability condition. For practical purposes it is usually reduced to $|D'| < |S'|$. This provides us with a more interpretable stability condition -that the demand curve has less steep slope than the supply curve. On first glance, this property is satisfied after examination of Figure 3.2. For this case, competitive forces will cause the price to converge to P_E from any initial value of the market price. This convergence is indicated by the arrows along the price axis. On the other hand , functions depicted in Figure 3.3. violate this condition since demand appears to have less steep slope than supply, so the market price does not converge to P_E and the equilibrium (market) is unstable.

4.2. When solving the equation F(x) = 0 by means of iteration method, many modifications are at the disposal how to rewrite F(x) = 0 in an equivalent form f(x) = g(x) (mostly x = f(x)) in order to make the iteration process convergent. In economic applications as mentioned above such "benevolence" loses any sense. Obviously (3.3) may be rewritten in the form

$$(4.1)\ D^{-1}(P_k) = S^{-1}(P_{k-1}) .$$

It is obvious, that any different, but mathematically equivalent expression of (4.1) represents another market (but with the same equilibrium price).

4.3. As a consequence of $P_k \to P_E$ we get $| Q_S^k - Q_D^k | \to 0$ and $Q_S^k \to Q_E$, $Q_D^k \to Q_E$ as $k \to \infty$ (graphical interpretation in Figure 3.2). That means, when market price converges to equilibrium price, then the surplus and the shortage of good converge to zero. The whole process converges to the equilibrium point $[Q_E , P_E]$.

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