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ON THE NEGATION OF THE PREDICATES WITH APPLICATIONS IN COMBINATORIAL PROBABILITIES

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Abstract. Having as start point some predicates with finite and discrete domain, and using the negation rules, we try to eliminate the semantic ambiguities, through examples come from *Combinatorial Probabilities* problems. **MR classification:** 03A02, 05A05, 60A99.

Many times, in *Combinatorics* and in *Combinatorial Probabilities*, there are structures based on the expressions: „*at most*", respectively „*at least*" and which must be denied. Normally, these last expressions being not complementary, we want to study in the following, the correct manner of the denial and for them, we shall use the discrete *Probability Calculus*.

At the beginning, we recall the necessary notions (see [1] and [5]).

Definition 1. A *mathematics sentence* is an enunciation about which we say that it is true or false. ■

Example 1. i) „ $\sqrt{2} \in \mathbf{R} \setminus \mathbf{Q}$ " is a true sentence.

ii) „ $x \geq y,\ x, y \in \mathbf{N}$ " is an enunciation about which we can nothing conclude, hence it is not a *mathematics sentence*. ■

Definition 2. The *predicate* or the *sentence with variables* is an enunciation which depends on one, or many variables, with the property, that for every value of the variable, we obtain a true, or a false sentence. ■

Remark 1. i) Any predicate is a function $P : M \to \{f, t\}$, where *M* is a fixed set, the value domain being represented by *false* and *true*.

ii) The essential difference between *sentence* and *predicate* can be seen the best, in the enunciation of the *logical implication*. So, for the *sentences P* and *Q*, this is:

(1) $P(x) \to Q(x)$,

but for the *predicates* $P(x)$ and $Q(x)$, the implication becames:

(2) $P(x) \Rightarrow Q(x)$,

only when:

(3) $(\forall)x(P(x) \to Q(x))$ is true.

iii) The *equivalence of the predicates* $P(x)$ and $Q(x)$, noted:

(4) $P(x) \Leftrightarrow Q(x)$

appears when the sentence:

(5) $(\forall)x(P(x) \leftrightarrow Q(x))$ is true. ■

Example 2. $P(x)$: „*x is a complex number*" is obviously a *predicate*. ■

In *Combinatorics* and in *Combinatorial Probabilities*, we are interested in the predicates with domain *M*, discrete and finite. So, *M*, from the **Definition 2**, has the form:

$$M = \{x_1, \ldots, x_n\}, \text{ where } P(x_i) \in \{f, t\}, (\forall) i \in \{1, \ldots, n\}.$$

Then, we shall notice, that for the predicates with a finite domain *M*, the assertion:

„ $(\exists)x \in P(x)$" is reconsidered as „ $P(x_1) \vee \ldots \vee P(x_n)$", while:

„ $(\forall)x \in P(x)$" is reconsidered as „ $P(x_1) \wedge \ldots \wedge P(x_n)$".

More, we can rewrite them, in the equivalent form:

(6) $((\exists)x \in M, P(x)) \Leftrightarrow ((\exists)x((x \in M) \Rightarrow P(x)))$,

(7) $((\forall)x \in M, P(x)) \Leftrightarrow ((\forall)x((x \in M) \wedge P(x)))$.

In the following, we shall study the negation of a predicate $P(x)$, denoted $\rceil P(x)$, or $\overline{P(x)}$. The denial quantifier, applied to the implication (2), gives us any of the next two relations:

(8) $(\rceil(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\rceil P(x)) \vee Q(x))$,

(9) $(\rceil(P(x) \Rightarrow Q(x))) \Leftrightarrow (P(x) \wedge (\rceil Q(x)))$.

If we deny (6), with (8), it occurs:

(10) $(\rceil((\exists)x \in M, P(x))) \Leftrightarrow ((\forall)x((x \notin M) \vee P(x)))$.

while (9) conducts to:

(11) $(\rceil((\exists)x \in M, P(x))) \Leftrightarrow ((\forall)x((x \in M) \wedge \rceil(P(x))))$.

Denying now the relation (7), we have:

(12) $(\rceil((\forall)x \in M, P(x))) \Leftrightarrow ((\exists)x((x \notin M) \vee (\rceil P(x))))$.

From [3], we shall be in need of the:

Definition 3. i) In the classical meaning, the *probability of the event A*, denoted by $P(A)$, is the ratio:

(13) $P(A) = \dfrac{\text{success numbers}}{\text{possibility numbers}}$.

ii) The *conditional probability* of the event A, through the event B, is:

(14) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, when $P(B) > 0$.■

We go on our logical periplus on the *predicates*, by the help of some concrete cases, which appear in four *Combinatorial Probabilities* problems.

Problem 1. In a box, there are 100 screws, from which 10 are rejects. Simultaneously, we draw out 5 screws. We consider the next events:

A = {at least a screw from 5 is a reject},
B = {at most 2 screws from 5 are rejects}.

Show explicitly the form of $\overline{A}$ and $\overline{B}$, and then, compute their probabilities.

Solution. For a *good* or a *reject* screw which appears in our events A and B, we shall use the predicates $G(x)$ respectively $R(x)$, when $x \in \{x_1, \ldots, x_5\}$. With (6), it follows:

(15) $A : ((\exists)x((x \in \{x_1, \ldots, x_5\}) \Rightarrow R(x)))$,

and with (11), it holds:

(16) $\overline{A} : ((\forall)x((x \in \{x_1, \ldots, x_5\}) \wedge G(x))) \Leftrightarrow (G(x_1) \wedge \ldots \wedge G(x_5))$, therefore:

$\overline{A}$ = {all the 5 screws are good}.

Using now, the *hypergeometric scheme* (see [4]), we find:

$$P\left(\overline{A}\right) = \frac{C_{90}^{5}}{C_{100}^{5}} = 0{,}583.$$

Also, with (6), we have:

(17) $B : ((\exists)x((x \in \{x_1, x_2\}) \Rightarrow R(x)))$,

and again, applying it the relation (11), it holds:

(18) $\overline{B} : ((\forall)x((x \in \{x_1, x_2\}) \wedge G(x))) \Leftrightarrow (G(x_1) \wedge G(x_2))$.

With the *hypergeometric scheme*, we find:

$$P\left(\overline{B}\right) = \frac{C_{90}^{2}}{C_{100}^{5}} = 0{,}00005 \text{ .■}$$

Remark 2. We must notice, that for the denial of (15) and (16), the relation (10) is not at all useful ! ■

Problem 2. In a box, there are 100 screws, from which 10 are rejects. Successively, we draw out, without return, 3 screws. We consider the next events: A_i = {the screw drawn out, at the time i, is good}, $(\forall)i \in \{1,2,3\}$.

Sketch the following events, and then, compute their probabilities:

i) B = {all the screws drawn out are good};
ii) C = {at least a screw drawn out is good};
iii) D = {at least a screw drawn out is reject};
iv) E = {all the screws drawn out are reject};
v) F = {only a screw is good}.

Solution. As in the above **Problem 1**, we shall use the same predicates $G(x)$ and $R(x)$, for $x \in \{x_1, x_2, x_3\}$.

i) Obviously: $B : ((\forall)x((x \in \{x_1, x_2, x_3\}) \wedge G(x))) \Leftrightarrow (G(x_1) \wedge G(x_2) \wedge G(x_3)) \Leftrightarrow$

$\Leftrightarrow (A_1 \cap A_2 \cap A_3)$.

Because A_i, for every $i \in \{1,2,3\}$, are dependent events, with a well known relation (see [3]), we have:

(19) $P(B) = P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 / A_1) \cdot P(A_3 / (A_1 \cap A_2)) =$

$= \dfrac{90}{100} \cdot \dfrac{89}{99} \cdot \dfrac{98}{98} = \dfrac{2 \cdot 89}{5 \cdot 49} = 0{,}726$.

ii) Immediately, we have:

$$C : ((\exists)x((x \in \{x_1, x_2, x_3\}) \Rightarrow G(x))) \Leftrightarrow (G(x_1) \vee G(x_2) \vee G(x_3)) \Leftrightarrow$$
$$\Leftrightarrow (A_1 \cup A_2 \cup A_3).$$

Because A_i, for every $i \in \{1,2,3\}$, are dependent and compatible events, with the *inclusion – exclusion* or *Poincaré principle* (see [3]), we have:

$$(20)\ P(C) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) -$$
$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) +$$
$$+ P(A_1 \cap A_2 \cap A_3).$$

But:

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 / A_1) = \frac{90}{100} \cdot \frac{89}{99} = 0{,}809,$$
$$(21)\ P(A_1 \cap A_3) = P(A_1) \cdot P(A_3 / A_1) = \frac{90}{100} \cdot \frac{89}{89} = 0{,}809,$$
$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3 / A_2) = \frac{90}{100} \cdot \frac{89}{99} = 0{,}809.$$

So, with (19) and (21), in (20), it holds:

$$(22)\ P(C) = P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{90}{100} - 3 \cdot 0{,}809 + 0{,}726 = 0{,}999.$$

iii) We notice that:

$$D : ((\exists)x((x \in \{x_1, x_2, x_3\}) \Rightarrow R(x))) \Leftrightarrow (R(x_1) \vee R(x_2) \vee R(x_3)) \Leftrightarrow$$
$$\Leftrightarrow (\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3).$$

Because $D = \overline{B}$, its probability is:

$$P(D) = 1 - P(B) = 0{,}274.$$

iv) The event E can be written:

$$E : ((\forall)x((x \in \{x_1, x_2, x_3\}) \wedge R(x))) \Leftrightarrow (R(x_1) \wedge R(x_2) \wedge R(x_3)) \Leftrightarrow$$
$$\Leftrightarrow (\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3).$$

Because $E = \overline{C}$, its probability is:

$$P(E) = 1 - P(C) = 0{,}001.$$

v) The form, of the event F, is:

$$F : ((\exists)x, (\forall)y((x, y \in \{x_1, x_2, x_3\}) \Rightarrow (G(x) \wedge R(y)))) \Leftrightarrow ((G(x_1) \wedge R(x_2) \wedge R(x_3)) \vee$$
$$\vee (R(x_1) \wedge G(x_2) \wedge R(x_3)) \vee (R(x_1) \wedge R(x_2) \wedge G(x_3))) \Leftrightarrow ((A_1 \cap \overline{A}_2 \cap \overline{A}_3) \cup$$
$$\cup (\overline{A}_1 \cap A_2 \cap \overline{A}_3) \cup (\overline{A}_1 \cap \overline{A}_2 \cap A_3)).$$

We notice that the events $A_1 \cap \overline{A}_2 \cap \overline{A}_3$, $\overline{A}_1 \cap A_2 \cap \overline{A}_3$ and $\overline{A}_1 \cap \overline{A}_2 \cap A_3$ are incompatible, because every 2 of them have nothing in common, for their intersection. Therefore, the probability $P(F)$ could be computed, using the formula:

$$P(F) = P(A_1 \cap \overline{A}_2 \cap \overline{A}_3) + P(\overline{A}_1 \cap A_2 \cap \overline{A}_3) + P(\overline{A}_1 \cap \overline{A}_2 \cap A_3).$$

For it, firstly, we have: $P(A_1 \cap \overline{A}_2 \cap \overline{A}_3) = P(A_1) \cdot P(\overline{A}_2 / A_1) \cdot P(\overline{A}_3 / (A_1 \cap \overline{A}_2)) = \frac{90}{100} \cdot \frac{10}{99} \cdot \frac{9}{98} = 0{,}008,$

$$P(\overline{A}_1 \cap A_2 \cap \overline{A}_3) = P(\overline{A}_1) \cdot P(A_2 / \overline{A}_1) \cdot P(\overline{A}_3 / (\overline{A}_1 \cap A_2)) = \frac{10}{100} \cdot \frac{90}{99} \cdot \frac{9}{98} = 0{,}008,$$
$$P(\overline{A}_1 \cap \overline{A}_2 \cap A_3) = P(\overline{A}_1) \cdot P(\overline{A}_2 / \overline{A}_1) \cdot P(A_3 / (\overline{A}_1 \cap \overline{A}_2)) = \frac{10}{100} \cdot \frac{9}{99} \cdot \frac{9}{98} = 0{,}008,$$

and so, it holds:

$$P(F) = 3 \cdot 0{,}008 = 0{,}024. \blacksquare$$

Problem 3. *A* urn contains *n* white balls. Successively and without return, we draw out at most *n*-1 balls and we denote this event, by *A*. Find the probability $P(\overline{A})$.

Solution. Because „*at most n-1balls*" means or 0, or 1,…, or *n*-1, we must notice that:

$$A:((\exists)x((x\in\{x_0,\dots,x_{n-1}\})\Rightarrow W(x))),$$

where $W(x)$ is predicate which corresponds to a white ball. From this, it follows:

$$\overline{A}:((\forall)x((x\notin\{x_0,\dots,x_{n-1}\})\vee W(x)))\Leftrightarrow W(x_n)\Leftrightarrow\{x_n \text{ is a white ball}\}.$$

Now, we know that $\overline{A}$ is equivalent with the fact that the *n* ball, drawn out from the urn, is white. So, keeping account that the process is without return, after the moment when we drew out from urn, *n*-1 balls, a only white ball remains. For it, the drawing out probability is:

$$P(\overline{A})=\frac{1}{1}=1.$$

At the same result, we could arrive with a property (see [4]) according to, if, in a urn with *a* white and *b* black balls, we draw out, *k* of them, without knowing their colour, then, the probability for the ball $k+1$, drawn out, to be white, is:

$$P=\frac{a}{a+b}.$$

In our case, $a+b=a=n$ and therefore, $P(\overline{A})=1$.■

Remark 3. The relation (11) is not useful solving this problem ! ■

Problem 4. O urn has *a* white and *b* black balls. Simultaneously and without return, we draw out $n\le a+b$ balls. We denote by *A*, the event in which, at most $n-1$ balls, from the anterior *n*, could be white. Find the probability $P(\overline{A})$.

Solution. The event *A* supposes that, from *n* balls drawn out of the urn, as white, could be or 0, or 1,…, or *n* – 1 of them. So, it follows:

$$A:((\exists)x((x\in\{x_0,\dots,x_{n-1}\})\Rightarrow W(x))),$$

where $W(x)$ is the predicate corresponding to a white ball. With the relation (10), the complementary event $\overline{A}$, of *A*, becomes:

$$\overline{A}:((\forall)x((x\notin\{x_0,\dots,x_{n-1}\})\vee W(x)))\Leftrightarrow \text{(all the balls drawn out are white)}.$$

Similarly, using (11) and the predicate $B(x)$, associated to a black ball, we arrive at the same result, because:

$$\overline{A}:((\forall)x((x\in\{x_0,\dots,x_{n-1}\})\wedge B(x)))\Leftrightarrow \text{(all the balls drawn out are white)}.$$

The probability $P(\overline{A})$ will be immediately compute, through the *hypergeometric scheme* and:

$$P(\overline{A})=\frac{C_a^nC_b^0}{C_{a+b}^n}=\frac{C_a^n}{C_{a+b}^n}.■$$

Remark 4. In [2], the above **Problem 4** was a principal idea for the unusual proof of a very well known *Combinatorial identity*:

$$\sum_{k=0}^{n}C_a^kC_b^{n-k}=C_{a+b}^n,\ \text{when } a,b,n\in\mathbf{N}, a\ge k, b\ge n-k, a+b\ge n.■$$

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