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# A "Big Problem" as a Motivator and Backbone For Mathematics Learning

**Dan Fendel,** San Francisco State University, **Nitsa Movshovitz-Hadar,** Technion – Israel Institute of Technology**, Diane Resek,** San Francisco State University

## Introduction

In many countries, high school mathematics consists of chapters in Algebra, Geometry, Trigonometry, and Pre-Calculus organized as a curricular sequence. Calls for an integrated curriculum began as early as the 1960's. One early response was Fehr's Unified Modern Mathematics from Columbia University, but that program was intended for the top 10% of the population. This paper describes IMP—Interactive Mathematics Program—a four-year problem-centered curriculum, developed in the United States for implementation in heterogeneous high school classes[1]. This curriculum program is intended to replace the traditional program of Algebra, Geometry, Trigonometry and Pre-Calculus. It consists of five learning units for each of the four years. Every unit develops from a "Big Problem" that serves as the motive and backbone for the development of relevant mathematics concepts and skills. Concepts arise and skills are practiced through a variety of smaller problems, explored and solved over a period of 6-8 weeks (with the equivalent of five 50-minute math periods a week), thus leading to the solution of the Big Problem. Some of these central problems are based in practical real-world situations, such as maximizing profits for a business or studying population growth. Others are more fanciful, involving situations like a baseball pennant race or a circus act. They may have connections with human history, various sciences, or classic literature, providing students with a view of how mathematics problems actually evolve from a variety of circumstances.

This paper discusses two sample units—*Do Bees Build It Best?* (a Year 2 unit) and *Orchard Hideout* (a Year 3 unit)—and the gains of the problem-based approach. It also includes a short description of other pedagogical features of the program, including grouping students to work collaboratively in small groups and fostering student communication through both written and oral presentations. Finally, there are comments on student achievement in the program.

## Two sample units[2]

### *Do Bees Build It Best?* (Year 2 unit)

In this unit, students are presented with the following problem: *Bees store their honey in honeycombs which consist of cells they make out of wax. What is the best shape for a honeycomb cell?*

In the course of their work on this problem, students learn about area, volume, the Pythagorean theorem, and the isoperimetric problem. In the opening days (days 1-2) of the unit, after initial concrete exploration of ideas of volume, the central problem is restated as an issue of "efficiency": *How can bees get the most storage space out of their wax?*

Students begin their work on this problem by studying the general idea of area, including the concept of a unit of measurement. Early in this investigation, they revisit and then build on the principle that the area of a rectangle can be found by multiplying its length and width. Using the geoboard as a tool, they focus their investigation next on triangles, and discover the formula for the area of a triangle (days 3-8). This early work with area emphasizes concrete experiences and the idea that area is basically "counting units." Students also find area formulas for parallelograms and trapezoids.

After a brief review of right triangle trigonometry, which they had learned in a Year 1 unit called *Shadows,* students discover the Pythagorean Theorem by comparing the areas of the squares constructed on the sides of a right triangle (days 9-10). They then prove the Pythagorean theorem using an area argument and apply it in a variety of situations (days 11-16).

With the tools of area measurement, right triangle trigonometry, and the Pythagorean theorem in hand, students return to the unit problem, first looking at the issue of "efficiency" in terms of area and

[1] In October 1999, the IMP curriculum was designated by the U.S. Department of Education as an "Exemplary" mathematics program. This was the highest possible rating, given to only 5 of over 60 programs reviewed.
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[2] Fendel and Resek were authors of IMP. Movshovitz-Hadar contributed by developing the conceptual framework and outlines for the two sample units described in this paper.

perimeter. Through a series of problems, they explore what polygon gives the greatest area for a given perimeter (days 17-20). Initially, they look at rectangles and see that squares are optimal. Generalizing from that example, they focus on regular polygons, and develop a formula for the area of a regular polygon of a given perimeter in terms of the number of sides. Students discover that the larger the number of sides, the greater the area. The circle is viewed as a limiting figure for the sequence of regular polygons having a fixed perimeter. Day 21 is devoted to notation and terminology for inverse trigonometric functions, which arise in the earlier work.

Students next turn their attention to volume, using a concrete approach similar to that used with area. They focus specifically on right prisms, examining relationships among various measurements - the volume, the lateral surface area, the area of the base, the perimeter of the base, and the height (days 22-27). Through this investigation, they find that the question

*Which prism has the largest volume for a fixed height and lateral surface area?*

reduces to the question *Which base for a prism gives the largest area for a given perimeter?*
Using their earlier findings on regular polygons, students see that cylinders might be candidates for the optimal prism. The cylinder is rejected, however, as a candidate for the shape of the honeycomb, because cylinders cannot be fitted together without leaving gaps. That is, they do not *tessellate*. With this added condition that the figures need to tessellate, the final refinement of the unit question becomes *Among prisms whose base is a regular polygon that tessellates, which one will have the largest area of the base for a fixed perimeter of the base?*

Students find that cells in the shape of regular hexagonal prisms, the choice of the bees, is the mathematical winner. This synthesis of ideas brings up the solution of the unit problem (days 28 and 29). The last two days are devoted to assessments and summing up. There is an option at this point to conclude the unit by having students watch the video, "The Mathematics of the Honeycomb."

Concepts of measurement—especially area, lateral surface area, and volume—are the mathematical focus of this unit. The main concepts and skills that students encounter and practice during the course of the unit can be summarized by category as follows:
Area: Understanding the role of units in measuring area; Recognizing that the perimeter of a figure does not determine its area; Discovering formulas for the area of rectangles, triangles, parallelograms, and trapezoids; Establishing that a square has the largest area of all rectangles for a fixed perimeter; Developing a formula for the area of a regular polygon of a given perimeter in terms of its number of sides; Discovering and demonstrating that, for a fixed perimeter, the more sides a regular polygon has, the greater its area; Discovering and demonstrating that the ratio of the areas of similar figures is equal to the square of the ratio of corresponding linear dimensions.
Pythagorean theorem: Discovering the Pythagorean theorem by comparing the areas of the squares constructed on the sides of a right triangle; Proving the Pythagorean theorem using an area argument; Applying the Pythagorean theorem in a variety of situations.
Surface area and volume: Understanding the role of units in measuring lateral surface area and volume; Developing principles relating the volume and lateral surface area of a prism to the area and perimeter of its base; Discovering that the ratio of the lateral surface areas of similar solids is equal to the square of the ratio of corresponding linear dimensions, and that the ratio of the volumes of similar solids is equal to the cube of the ratio of corresponding linear dimensions.
Trigonometry: Reviewing right triangle trigonometry; Finding the range of the basic trigonometric functions (for acute angles); Using the notation and terminology of inverse trigonometric functions; Applying trigonometry in a variety of problem-solving settings.
Miscellaneous: Reviewing similarity; Strengthening two- and three-dimensional spatial visualization skills; Examining the concept of tessellation and discovering which regular polygons tessellate; Developing the general concept of an inverse function.

***Orchard Hideout* (Year 3 unit)**

The central problem of this unit concerns a couple who have planted an orchard of trees in careful rows and columns on a circular lot. Early in the unit (days 1-4), students see the usefulness of representing the orchard using the coordinate system, and use this system to represent the orchard design by placing a tree at every lattice point within the lot (these are points with integer coordinates) except the origin.

The couple in the unit problem realize that after a while the trunks of their trees will become so thick that they will no longer be able to see out from the center of the orchard. In other words, the orchard will become a "hideout." The main question of the unit is *How soon after they planted the orchard would the center of the lot become a true "orchard hideout"?*

Students' search for the answer to this question leads them to the study of several aspects of circles and coordinate geometry. In order to use the information that the couple has about the growth rate of their trees, students need to understand how the circumference, area, and radius of a circle are related to each other (days 11-15). They begin by comparing circles to circumscribed squares and use principles of similarity to see two key facts:

•The circumference of a circle is some fixed number (the "circumference coefficient") times its radius.

•The area of a circle is some fixed number (the "area coefficient") times the square of its radius.

By looking at various other circumscribed regular polygons, students gain further insight into the formulas for area and circumference of circles and the connection between the two. In particular, they see, both through polygon examples and through an area analysis, that the circumference coefficient is exactly twice the area coefficient. This observation leads to the definition of $\pi$ and to the formulas for circumference and area: $C = 2\pi r$ and $A = \pi r^2$.

Another key component of the unit is finding the distance from a given lattice point to a given line of sight out of the orchard. To do this, students need to develop and combine a variety of ideas from coordinate geometry, synthetic geometry, and right triangle trigonometry. Their work with coordinate geometry builds on the Pythagorean theorem and includes development of both the distance formula and the midpoint formula (days 6-8). A key idea from synthetic geometry is the principle that a line through the midpoint of a line segment is equidistant from the segment's endpoints. Students develop and prove this principle through an activity (days 9-10) that grows out of one of the unit's Problems-of-the-Week (abbreviated POW, see details below). The actual computation of distance from a point to a line is done in several ways, using both similarity and right triangle trigonometry, thus building on students' work in Year 1 and Year 2 (days 16-18).

Toward the end of the unit, students need to determine which line of sight will remain unblocked the longest (days 20-21). Their search for the last line of sight begins with a detailed study of distances to lines of sight in the orchard of radius 3. They see that the last unblocked lines of sight are those closest to the coordinate axes. They then "eyeball" lines of sight for the orchard of radius 6, and their intuition suggests that this principle regarding the location of the last line of sight should be true in general. They are told that this intuition is correct, but that a proof of this fact is quite advanced, and requires concepts that they have not yet studied at this point in their mathematical careers.

The unit ends with students working in groups to develop a complete solution to the unit problem, finding both the distance to the last line of sight from its nearest lattice point and the time required for the couple's trees to reach the necessary size, based on data provided on the growth of the cross-sectional area of the trees (days 22-23).

Along the way through the solution of the unit problem, students encounter a variety of tangents. (This reference to tangents is both figurative and literal.) One of their results is a proof that a tangent to a circle is perpendicular to the radius at the point of tangency (day 10). As a sidelight to their work with the distance formula, students develop the general equation of a circle (day 5). They later use the technique of completing the square to put certain quadratic equations into standard form to find the radius and center of the circles they represent (day 19).Other ideas arise through the unit's POWs. For example, students prove basic facts about perpendicular bisectors (days 8-10) and angle bisectors (day 15), thereby establishing the existence of both circumscribed and inscribed circles for triangles.

Over the course of this unit, students develop a wide range of ideas from geometry and algebra. The main concepts and skills that students encounter can be summarized as follows:

<u>Coordinate geometry</u>: Using the Cartesian coordinate system to organize a complex problem; Developing and applying the distance formula; Developing the standard form for the equation of a circle with a given center and radius; Finding the distance from a point to a line in a coordinate setting; Developing and applying the midpoint formula.

<u>Circles</u>: Using similarity to see that the circumference of a circle should be some constant times its radius and that the area of a circle should be some constant times the square of its radius; Finding formulas for the perimeter and area for regular polygons circumscribed about a circle; Using circumscribed polygons to see that the "circumference coefficient" for the circle is twice the "area coefficient" for the circle;

Defining $\pi$ and seeing why it appears in both the formula for circumference and the formula for area of a circle; Developing and applying the formulas for the circumference and area of a circle.
Synthetic geometry: Identifying and describing a set of points satisfying a geometric condition; Discovering and proving that the set of points equidistant from two given points is the perpendicular bisector of the segment connecting the given points; Defining the distance from a point to a line and proving that the perpendicular distance is the shortest; Discovering and proving that any line through the midpoint of a segment is equidistant from the endpoints of the segment; Discovering and proving that the set of points equidistant from two intersecting lines is the union of the bisectors of the angles formed by the lines.
Algebra: Using the technique of completing the square to transform equations of circles into standard form; Using algebra in a variety of proofs involving coordinates and angles.
Logic: Understanding and using the phrases "If . . . , then . . . " and "if and only if" in definitions and proofs; Working with converses.
Miscellaneous: Using symmetry to help analyze a problem; Learning about Pythagorean triples.

## The Gain from Problem-Based Curriculum

The richness of the mathematics in the two problem-based units just discussed is apparent from the descriptions. The curricular principle of embedding this rich mathematics in context has several advantages.

One key feature of this approach is that it demonstrates to students that mathematics is a living tool, with relevance in many areas. The traditional student cry, "When will I need this?" is gone, because students see mathematics growing out of applications. Moreover, students find the contexts intrinsically engaging, so the Big Problems serve as good motivators for student learning.

Another major gain is that context provides opportunities for making connections. For example, in *Orchard Hideout,* students see the connection between algebraic ideas, such as completing the square, and geometric ideas. They combine trigonometry with study of the coordinate system. They learn fundamental elements of mathematical logic in the setting of work that is primarily geometrical. Other IMP units make other connections, both within and beyond mathematics. For instance, *The Pit and the Pendulum* uses the classic Poe short story as the setting to study standard deviation as well as principles of curve fitting. *Meadows or Malls?* (Year 3) uses a problem about land-use planning as the context for work with both matrix algebra and the geometry of intersecting planes. In *High Dive* (Year 4), a circus act provides the context for analysis of the physics of falling objects, and principles related to constant acceleration, as well as the extension of trigonometric functions beyond the right-triangle setting.

## The Pedagogical Approach

This four-year curriculum includes the fundamental ideas, concepts and skills from Algebra, Geometry, Trigonometry and Pre-Calculus traditionally found in most countries' curricula, and in addition some less traditional topics such as linear programming and statistical inference. In addition to the principle of organizing the curriculum around Big Problems, the program has these important pedagogical features:

### *Student Text*

IMP engages students in active problem solving by assigning students the following main types of assignments:

(i) In-class activities, devoted to brainstorming of new ideas and exploration of new concepts, using peer communication as resources for consultation and for bouncing their ideas, and teacher-student communication for guidance and support. Students work collaboratively in randomly formed small groups, attempting these activities. They develop their knowledge experientially through exploring the need for math to solve a problem, gathering data, looking for a pattern, investigating a conjecture, convincing one another of the findings, and so on.

(ii) Daily homework through which class work is reinforced on an individual basis, and extended towards the next class meeting. Some of the homework assignments provide opportunities for students to reflect upon the work the have done over a longer period of time, and find new connections among topics seen as disconnected at first sight.

(iii) POWs – Problems of the Week. These are longer-term projects, mainly open-ended problems that require time and peace-of-mind to elaborate. These also serve as a vehicle for improving students' ability to reason mathematically and to express in writing mathematical ideas. They also provide opportunities for oral presentation of mathematical ideas. POWs are not necessarily connected in a direct way to the

unit they are embedded in, but in many cases they are. (See example in the discussion of *Orchard Hideout.*)
(iv) Supplemental Problems. These problems are used as a tool for tailoring the curriculum to heterogeneous classes by assigning them in different ways to different students according to their needs. Some can be used to reinforce skills and basic concepts and others can be used for extension of ideas beyond the basic curriculum.

***Classroom Inquiry and the Teacher's Guide***

The teacher's role in IMP class is that of a facilitator for the learning process, a process that is time consuming and takes a lot of patience. Quite often teachers must hold back in order to let things happen. Other times the teacher provokes a dilemma, poses a question that promotes thinking, asks a leading question to focus a student's attention to a missing point, or asks a testing question to see where a student is going with an idea. As a facilitator of learning, teachers are responsible for allowing a significant amount of class-time to student reports and whole class discussions. The discussions capitalize on students' individual work and on ideas develop during group work. Whole class discussions constitute the forum in which knowledge is shared, crystallized and become public knowledge, as a result of bouncing back and forth ideas student initially construct on their own during their work.

The teacher's guide to a unit is a series of daily lesson plans, the core of which are students' activities and discussions. The teacher's guide includes suggestions for various questions the teacher can ask to stimulate student thinking, as well as mathematical background beyond what students are to learn from the unit, and specific grading suggestions for student's achievement evaluation.

***Assessment and Grading***

On-going assessment and occasional grading are necessary components of every educational program. Without them, neither the teacher nor the students can become aware of the progress made through the educational process and of changes that might be needed in order to achieve the goals of the program. IMP's recommendation to teachers is to evaluate student work in a multifaceted way, using a variety of criteria, including daily work such as oral presentations, written work on various assignments, contribution to group projects, participation in class discussion, end-of-unit performance test, and end-of-unit personal portfolio and self-assessment submitted by each student upon completion of a unit.

## Evidence of Student Achievement in the Program

IMP was initiated to address the challenge of NCTM agenda for the 90's, which set standards to decompartmentalize the mathematics curriculum and change the culture of the mathematics classroom. It was unknown at the time whether these standards were actually attainable. IMP was one of five secondary school programs whose development was supported by the U.S. National Science Foundation. IMP's development was overseen by an advisory board including reputable mathematicians and mathematics educators. The four years of IMP curriculum were pilot-tested in the early 90's in several states. The pilot testing was followed by several rounds of review, revision, and further field testing.

Evaluation of student achievement in IMP has been extensive. Studies show that IMP students do significantly better than students in traditional programs in areas such as quantitative reasoning, statistics, and general problem solving. Repeated evaluations demonstrate that while learning more in those areas, IMP students also score at least as well on standardized tests as students from traditional programs. IMP students have been admitted to first-rate colleges throughout the United States, and been successful in their academic endeavors. In addition, a study showed that IMP students tended to take more years of high school mathematics, when the study was optional, than did their peers in traditional programs. For more information on the evaluation of IMP, go to
http://www.mathimp.org/research/index.html

## References

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