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Perception of infinity: does it really help in problem solving?

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Some starting points

The basic objects in mathematics are the numbers and the set of numbers. As mathematics is dealing with infinite set of numbers and as those are the only infinite objects that we "know" so far, a lot of questions connected with the idea of infinity could be addressed to school learning. Does an intuition of infinity really exist in children? Could it be stimulated in school? Does the school build an understanding of infinity? How is this understanding reflected on problem solving along the schooling period? These are some of the questions the study attempts to answer.

According to Fischbein (1987), intuitive knowledge is a self-explanatory cognition that we accept with certainty as being true. It is a type of immediate, coercive, self-evident cognition, which leads to generalizations going beyond the known data. Fischbein differentiated between primary intuitions and secondary intuitions. Primary intuitions were defined as intuitions that "develop in individuals independently of any systematic instruction as an effect of their personal experience" (Fischbein, 1987, p. 202). Secondary intuitions were defined as "those that are acquired, not through natural experience, but through some educational intervention" (Fischbein 1987, p. 71). Secondary intuitions were defined as evident when formal knowledge becomes immediate, obvious, and accompanied by confidence. Secondary intuitions about a certain concept or process are often inconsistent with the related primary intuitions about the same concepts.

The research findings in the literature connected to this topic indicated that the methods the students apply for the comparison of infinite sets were largely influenced by methods they had used when comparing finite sets. They usually did not use 1:1 correspondence, the criterion that should be used to determine the equivalence of two infinite sets within Cantorian set theory (Tall, 1990; Tsamir, & Tirosh, 1994; 1999; Tsamir 2002). It was also found that when students used more than one method for comparing infinite sets, they reached contradictory conclusions, of which they were usually unaware (e.g., Tirosh & Tsamir, 1996; Tsamir & Tirosh, 1999). These conclusions were drawn by assessing abilities of high school and undergraduate students.

The research

To identify primary and secondary intuitions connected to the idea of infinity, we considered important to cover a broad range of ages. The participants in our study were students from grades 4 to 12 (10-11 to 18-19 years old) and undergraduate students – prospective mathematics teachers; 172 students answered to questionnaires, and 21 students were interviewed. At the same time, interesting remarks have been generated by a two hours interactive discussion in a class of 30 students in grade 5 (11/12 years old). As we were interested to stimulate through open questions divergent opinions in all the three categories used for collecting data, and, consequently, the range of answers were very large, our study is

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focused on a qualitative analysis. In the present paper, we comment on the reactions of children 10 to 14 years old.

Children's everyday meaning for infinity

The open-ended question: "Describe the idea of infinity in your own words" received various answers that could be classified as follow:

- emphasizing **a temporal** dimension (e.g. Bissan (grade 8): *Infinity is something that never stops. It will go on and on forever.*)
- emphasizing **a spatial rhythmic** dimension (e.g. Rebecca (grade 8): *Infinity is when something doesn't finish and it keeps on going and going and never ends*.)
- emphasizing **affects and spirituality** (e.g. Octavian (grade 4): *Infinity is something that we can't grasp the secret. Our mind is bind and we can't say many things. It is not able to understand everything about infinity. This is a word that is endless in numbers, and love, etc. But not everything is endless. We can't get this secret but with the help of God. He can help us find the key to understand infinity. With the help of people we can't get this marvellous mystery of understanding with the help of people. Not even the greatest scientists can understand this mystery. It is only God Who can uncover this. It is only when we get in heavens that we can fully understand infinity.*)
- emphasizing **a processional** dimension (e.g. Xena (grade 8): *Infinity is the term we use for something that does not stop; it continues to rise.*). These processes are seen in terms of **change** (e.g. Bissan (grade 8): The number of desks in a classroom is considered finite because they have an amount that can't change in the same sense as an infinite number; or Anca (grade 8): *The set of the divisors of 32561784937289463785 is not infinite because there is only a number of numbers that never changes*.); or in terms of **numbering** (e.g. Xim (grade 8): *Finite is like the number of pencils in a room, but infinite is like numbering all numbers in the world.*)
- emphasizing **accountability** (e.g. Lie (grade 8): *Infinity is like all sounds in the world, which we cannot count*; or Kira (grade 8): *Infinity is like a never ending number. There is not a number that you can write because it does not finish*.).

Intuitions in understanding infinity

Children as young as 11 years old are able to explain that there is not a biggest natural number and this is why the set of natural numbers is infinite. (e.g. Andreea (grade 5): *There are an infinity of natural numbers because if I pretend that I found the biggest, I can add 1 and I get a bigger one*. Or Alice (grade 6): ***N** is infinite because we can count 1,2,3 and anywhere we arrive, we know that it is still going on.*)

In addition, children are able to make a transfer of reasoning from ***N*** to ***Q***. The following discussion between the interviewer and a girl in grade 8 underlie this transfer.

Interviewer: What do you think, between 2 and 5 are more than 100 rational numbers?
Ana-Maria (grade 8): Er…*we could know how many numbers are between 2.01 and 3 and then…*
I: You said 2.01; is it the smallest fractional number from this interval?
A: *No, it is not the smallest…*
I: Can you tell me another one, which is smaller, in this interval?
A: *Yes, 2 point 0000000000…1…*
I: Oh, I understood…So…
A: *But it is not the smallest…*
I: Even this is not the smallest?
A: *No…*
I: Then, which is the smallest?

A: *Well, it is 2 point…very many zeros…and 1…*
I: And if you say 1000 of zeros, and then 1, is this the smallest number?
A: *Not necessarily…*
I: Can you tell a smaller one?
A-M: *Well, I can't …*
I: Why?
A-M: *Because …there are an infinity of numbers like this …*

Other children also had been able to bring evidences for the infinity of given sets by using arguments specific to the sequence of natural numbers. Some examples are quoted below.

Xim (grade 8): ***N*** *is infinite because we can't write the biggest number. The same, in the interval (2, 3), we can't write all the decimals of numbers, so, this set is infinite.*

A sequence in an interview with a sixth grade girl focus on the same idea:
Interviewer: Is the set of rational numbers between 2 and 5 infinite?
Alice (grade 6): *Yes!*
I: Why? Look, I have the smallest number and the biggest… why should be this an infinite set?
A: *Well, yes, could be 2.1; 2.11 … I mean 2 point … 111 and so on, I mean …*
I: Are these rational numbers?
A: *… they could be written as fractions…*
I: So, they are rational numbers … and you say they are an infinity…
A: *Perhaps they are not quite an infinity, because finally we still get to number 3, but they can be said as a sequence … it might be … number 2.1., it might be 2.11 plus 2.19, and so on … number 2.11 might be 2.111 and 2.119 … and so on…*

Other two answers to the same question:
Madalina (grade 5): *After 2 we could add as decimals all the natural numbers. In this way we can associate:*

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2.0 & 2.1 & 2.2 & 2.3 & 2.4 & 2.5 \end{array}$$

Bissan (grade 8): (2;5) *is an infinite set because there could be an infinite amount of decimals after the decimal point."*

Contrary to other studies, we found that, for many children in the lower secondary school, the concept of infinity of ***N*** is strong enough to permit reasoning. For example, some students found necessary to apply the negation to argue that a given set is finite. Some quotations:
Rebecca (8): *The divisors of 24 are not infinite, nor the divisors of 325617849372894637 85, because the last divisor is the number itself and there are bigger numbers than them.*
Vrit (8): *The biggest fractional number in the interval* (2;5) *is 4.9…., no, it doesn't exists because it is an infinite set.*

Particularly, we have been surprised by the accuracy in reasoning of some students in grade 5. During the discussion with the fifth graders, they effectively found rules to associate the following peers of sets: {1;2;3;… } and { -1;-2;-3; … }; {0;1;2;3;4; … } and {0;2;4;6;8; … }; {0;2;4;6; … } and {1;3;5;7; … }. First, they easily and readily identified the generating pattern of each sequence. They had some difficulties in finding the connections between the sets in each peer, but finally they discovered by their own the associative rules. Later on, the question *Which of the sets*: {0;3;6;9;12;15;…} *or* {1;2;4;5;7;8;…} *has more elements?* generated strong controversial discussions. The first reaction was reasoning on the finite case: from 0 to 15, the first set has 6 elements and the second has 10 elements, because, successively distributing the elements from **N**, one element enters the first set and two elements enter the second. Within this type of argument, the majority of children agreed that the second set has two times more elements than the first. Yet, some students disagreed and started to ask themselves how to proof/demonstrate one or the other of the assumptions. In what was followed, three of the students' interventions were fundamental:

- Roxana: *There are no rules, so they cannot be compared.*
- Anca: *At the beginning, the numbers in the previous sets looked to be arbitrary (at random), then we found a rule, so, if we do not have the rule yet, this does not mean it doesn't exist*. Her reaction was completely spontaneous and unexpected within the discussion.
- The third spontaneous reaction was the one of a girl who discovered a rule:

Madalina: *I noticed that* 2x2 – 1 = 3; 2x4 – 2 = 6; 2x5 – 1 = 9; 2x7 – 2 = 12.
It is to observe that in the previous examples the associative rules started from the first set, but here the starting point is the second set. Madalina made an implicit transfer; it looks like for her the 1:1 correspondence between the two sets in each peer was obvious. Doubting about the rule, one boy argued that it is not good for 0. Another boy remarked the formula 2x1 – 2 = 0.

Therefore, fifth graders in an ordinary state school in Bucharest were able to deal with numerable infinity, finding unexpected solutions. The debate stimulated their intuition and made them to connect intuitions with their knowledge. It seems that the intuition of infinity is already constructed in a 10 years old mind and an open atmosphere and targeting discussions could reveal this intuition. Identifying this intuition and using it in training could conduct to a deeper understanding of the numerical sets and the operations with numbers (Singer, 2002). In addition, it could open the way for a deeper understanding of the material and immaterial world.

Misconception: extrapolating from finite to infinite

Nevertheless, for many children, the misconceptions reported in the literature are present. Some of these misconceptions are exemplified below. As other studies reported (Tsamir, 2002), we found evidence of the continued influence of intuitive ideas associated with finite sets, which interfered with the students' ability to reflect on their judgments and affected their awareness of possible contradictions. The following questions permitted insights into the way of thinking about the equivalence of infinite sets:
Which of the sets {2;4;6;…} and {1;2;3;4;…} has more elements?
Which of the sets {0;2;4;6;…} and {1;3;5;7;…} has more elements?
A representative example for this category of approaches is the one of Michael's:
Michael (8): *I guess the positive integers are more than the even numbers because if you number till 10 there are more integers then even numbers*.

To discern which set has more elements, the children used intuitively various types of arguments. Frequently, they used **to count in N, with a given start and step** (e.g. Kurt (grade 8): *The positive integers are more because they are two times more than the even numbers;* Eren (grade 8): *Positive numbers {0;1;2;3;… } are more than {0;2;4;6;…} because they don't go 2 by 2;* Ioana (grade 8): *There are the same number of odd and even numbers because we count from 2 to 2 for each set;* Kurt (grade 8): *The sets of odd and even numbers are equal because they have the same amount of elements.*)
Another method the students used is **the relationship whole-part** (e.g. Stepan (8): *The positive numbers have more elements because they contain the even numbers + additional ones*.; Xim (grade 8): *The set of even numbers has more elements because 0 is there*.; Meenakshi (8): *The set of even numbers has more elements because it contains one extra # 0 and odd numbers start from* 1.; Roxana (5) *A part of an infinite set is finite*.; Kurt (grade 8): ***N** has more elements than {–1; –2; –3;…} because it includes and starts from 0 but the second set doesn't include 0, it starts from –1*.)

Misconception: margins

Some other misconceptions could be associated with various beliefs connected with the margins of sets. Thus, some students think that ***finite is something that has margins.*** (e.g. Loredana (grade 8): (2,5) *is finite because starts at 2, and ends at 5*.; Ioana (grade 8): (1,2) *holds from 1 to 2*.; Xena (grade 8): *A segment is a finite set of points because it has 2 ends*.)

Connected to this, a frequent misconception is that ***infinity is something that has no margins.*** (e.g. Ana-Maria (grade 8): ***R*** *is infinite because we do not know which is the beginning and which is the end*.). What it is interesting with this girl is that she has the intuition of density on **R** when she is operating with geometrical entities:

Interviewer: In other words, my question is: how many points are on a segment? Draw a segment about 4 cm long. I drew one, look at it, and I put along it 60 points…how many points do you think there are necessary to fill in a segment?

A-M: *I don't know…*

I: But is it finite or infinite? You just said the segment is finite…

A-M: *Is finite but…*

I: So, there are a finite number of points on it?

A-M: *No.*

I: So, is it finite, or infinite?

A-M: *Well, it is finite, but …the number of points is infinite…*

I: How could you argue for this?

A-M: *Err…the points are at very small dimensions and … in a segment … we can't count them.*

Single infinity

A common perception is that all infinite sets are equivalent, all have the same number of elements; this is called on short "single infinity". The intuition of single infinity is better revealed – even if it happens that the answer is wrong – when dealing with geometry. One of the geometry questions was: "If from an axis I cut a segment 1 Km long, I get another axis. Which one is longer?" The following answers give insights into students perceptions.

Alice (grade 6): *Err… would be the same thing … the first axis should be longer because it has 1 Km in addition, but it looks to be the same thing with the set of rational numbers between 2 and 5 and as the set of rational numbers between 2 and 3, that means an infinity is in both…*

Yakub (grade 8): *It will be longer because the line never ends it keeps on going until the end so as much as you cut the line; it still goes; it is infinity that will never end.*

Kira (grade 8): *I think it will be the same because after you cut it you can go 1 Km more and you will never stop, it doesn't get less nor more.*

However, an intuition of infinity could also be connected with the awareness of doubts about the nature of infinity. For example, after the discussion with the fifth graders, presented below, in which only numerable sets have been used, a student addressed us an insistent question: *Are all the infinite sets of the same type?*

Final comments

This study showed that the primary intuitions of students are similar to those found among young students and prospective teachers reported in the literature. It also showed that specific training develop awareness in understanding how the idea of infinity works on numerable sets. Tsamir (2002) discussed prospective teachers' primary intuitions and their responses after two types of interventions, finding that the newly acquired skills are not strong enough to support secondary intuitions. For Fischbein, secondary intuitions are those that are completely in line with the formal theory. He explained, for instance, "if for a mathematician the equivalence between an infinite set and a proper subset of it becomes a belief – a self explanatory conception – then a new, secondary intuition has appeared" (Fischbein, 1987, p. 68). Our study suggests that maybe an earlier preparation, which builds on intuition in a very informal way, is needed.

Three findings could be resumed as results of this study. The first is telling us that some children, under appropriate conditions of training, are able to internalize a mathematical intuition of the infinite sets. This is happening as soon as they learn about the sets of natural and decimals number, and this is happening even at the age of 10-11 years old. The second finding refers to the fact that, when the students' arguments are consistent, they seem to be based on connections between algebraic and geometrical thinking. The third one shows that there is not a connection to age in the deepness of the intuition of infinity; the errors and misunderstandings are similar for various ages, as well as the correct reasoning related to the formal knowledge.

A conclusion that could be drawn from these findings is that, infinity being an important intrinsic concept connected with number formation, the aspects connected with the idea of infinity should be part of the training very early in concepts learning. We did not say as part of the curriculum because here any attempt to formalize is dangerous; on the contrary, the approach has to be made using various examples from different contexts and emphasizing various points of view, outside mathematics (Singer, 2002). Here, lot of precautions are necessary, because, on one hand, as we underlined at the beginning, only through numbers infinity could be defined and explained, and, on the other hand, the trap of paradoxes is always near by, when we are dealing with infinite sets. If we take into consideration recent researches in mind and brain, there is a close interrelationship between some natural predispositions-intuitions and the learning process, which rebuild connections and structures.

Further research, vertically developed, could compare the same children achievements at the age of 10-11 with the ones at the age of 13-14 and, eventually, later. The results of the study could be reflected on teaching-learning materials focused on developing the intuition of infinity.

References

Fischbein, E. (1987). *Intuition in science and mathematics*. Dodrecht, Holland: Reidel.

Singer, M. (2001). *Structuring the information – a new way of perceiving the content of learning*, Zentralblatt für Didaktik der Mathematik (ZDM)/International Reviews on Mathematical Education, MATHDI, 6, p.204-217

Singer, M. (2002). *New ways of developing mathematical abilities*, Proceedings of the 26th Conference of the International group for the Psychology of Mathematics Education

Singer, M. (2002). *L'éducation primaire en Roumanie - A la recherche des traditions perdues pour jouer sur les chances de l'avenir*, paper presented at the international meeting: *Qu'enseigne-t-on aujourd'hui en mathématiques dans les écoles élémentaires d'Europe et que pourrait-on y enseigner ?*, Paris, Grands Salons de la Sorbonne

Tall, D. (1990). *Inconsistencies in the learning of calculus and analysis*. Focus on Learning Problems in Mathematics, 12(3&4), 49-64.

Tirosh, D., Tsamir, P. (1996). *The role of representations in students' intuitive thinking about infinity*. Journal of Mathematical Education in Science and Technology, 27 (1), 33-40.

Tsamir, P., Tirosh, D. (1994). *Comparing infinite sets: intuitions and representations*. Proceedings of the 18th Annual Meeting for the Psychology of Mathematics Education (Vol. IV, pp. 345-352). Lisbon: Portugal.

Tsamir, P., Tirosh, D. (1999). *Consistency and representations: The case of actual infinity*. Journal for Research in Mathematics Education. *30* (2), 213-219.

Tsamir, P. (1999). *The transition from the comparison of finite to the comparison of infinite sets: Teaching prospective teachers*. Educational Studies in Mathematics. *38* (1-3), 209-234.

Tsamir, P. (2002). *Primary And Secondary Intuitions: Prospective Teachers' Comparisons Of Infinite Sets*, in CERME 2 Proceedings.