

Fibonacci and Some of His Relations

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Abstract: In this article we revisit the Fibonacci sequence and extend it in various directions. We contend this is a good topic to get people interested in recurrences and closed form representation. We discuss various Fibonacci related sequences and finally represent two third order recurrence relations in terms of binomial sums and hypergeometric functions.

Introduction

Leonardo Fibonacci was a mathematical innovator of the thirteenth century. He was born in Pisa, Italy and was also known as Leonardo Pisano, or Leonardo of Pisa. Fibonacci had a Moorish schoolmaster, who importantly introduced him to the Hindu-Arabic numeration system and computational methods. Abacus is a word that has had several meanings in the past. It is still used today to denote the Japanese and Chinese computing devices soroban and suan pan. Fibonacci uses it in a way peculiar to him, he gives it the sense of mathematics with the exclusion of Geometry. In certain contexts, he uses the word NUMERUS, that is, number, also in this sense. Thus 'liber abbaci' of 'liber de numero' simply means 'book of mathematics'.

Liber abbaci, written in 1202, was a book very ahead of its time, its level was not reached again until three hundred years later by Luca Pacioli's 'Summa de Arithmetica, Geometria, Proportioni & Proportionalita' (Venice, 1509), in which Pacioli writes explicitly that he resumes the work of Fibonacci.

Liber abbaci, was written by Fibonacci after widespread travel and extensive study of computational systems. In it, he explains the Hindu-Arabic numerals and how they are used in computation. This extremely famous book was instrumental in displacing the clumsy Roman numeration system and introduced methods of computation similar to those used today.

Fibonacci is today, particularly remembered for the sequence of numbers {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...} that is associated with the breeding of rabbits.

Rabbits

In the liber abbaci [1], Fibonacci introduced the following rabbit story which generated his famous sequence.

Suppose that

1. there is one pair of rabbits in an enclosure on the first day of January,
2. this pair will produce another pair of rabbits on February first and on the first day of every month thereafter; and
3. each new pair will mature for one month and then produce a new pair on the first day of the third month of its life and on the first day of every month thereafter.

The problem is to find the number of rabbits in the enclosure on the first day of the following January after the births have taken place on that day. Let **A** denote an adult pair of rabbits and let **B** denote a baby pair of rabbits, the following describes the numbers after one year.

Month	Number of A's	Number of B's	Total number of pairs
January	1	0	1
After births on first of			
February	1	1	2
March	2	1	3
April	3	2	5
May	5	3	8
June	8	5	13
July	13	8	21
August	21	13	34

September	34	21	55
October	55	34	89
November	89	55	144
December	144	89	233
January	233	144	377

Therefore the number of pairs of rabbits in the enclosure one year later would be 377. In general, from the last column, we may see that the number of pairs at any month is the sum of the two preceding entries which leads us to the idea of a recursion formula.

Recurrences and Sums

A sequence may be finite or infinite and may be designated by symbols such as

$$\{a_0, u_1, u_2, \dots, u_n, \dots\}.$$

A recurrence (or difference) equation is the discrete analog of a differential equation and may be represented by

$$f(n) - f(n-1) = y(n),$$

where n is a natural number.

The Fibonacci recurrence formula may be written as

$$F_n = F_{n-1} + F_{n-2}, \quad n > 2 \tag{R1}$$

with the initial values $F_1 = 1, F_2 = 1$.

We now briefly describe the following sequences and series, some of which will be useful later.

An ‘arithmetic’ sequence is $A := \{a, a + d, a + 2d + \dots\}$ here a is the first term and d is the difference between

terms. If $SA := \sum_{i=0}^n (a + id)$, then $SA := \frac{n+1}{2}(2a + nd)$.

A geometric sequence is $G_n := \{a, ar, ar^2, \dots, ar^{n-1}\}$. If $G_n = \sum_{i=0}^{n-1} ar^i$, where r is the constant ratio of two subsequent terms, then

$$G_n = \frac{a(1-r^n)}{1-r}.$$

For an infinite number of terms

$$\lim_{n \rightarrow \infty} G_n = \frac{a}{1-r} \quad \text{for } |r| < 1.$$

Binomial sums are important and are intrinsically connected to hypergeometric functions.

$$(x + y)^n = \sum_{r=0}^n \binom{n}{r} x^r y^{n-r}$$

is a Binomial sum, where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and } n! = n(n-1)\dots, 3, 2, 1.$$

If the ratio of two consecutive terms $\frac{T_{r+1}}{T_r}$, in a series $\sum_{r=0} T_r$, is a rational function of a positive integer r then we have a hypergeometric series

$${}_p F_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| z \right] = \sum_{r=0}^{\infty} \frac{(a_1)_r \dots (a_p)_r}{(b_1)_r \dots (b_q)_r} \cdot \frac{z^r}{r!},$$

where $T_0 = 1$ and $(a)_r$ is Pochhammer’s function defined by

$$\begin{cases} (a)_0 = 1 \\ (a)_r = a(a+1)\cdots(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)} \end{cases}$$

and $\Gamma(x)$ is the classical Gamma function.

Now we return to the Fibonacci recurrence and investigate some of its properties.

Properties

The Fibonacci sequence has many remarkable properties, moreover, it continues to find applications in many areas of science and mathematics.

Hagspihl [3] noted for the Fibonacci numbers 3, 5, 8, 13 that

$$\begin{aligned} (3 \cdot 13)^2 + (2 \cdot 5 \cdot 8)^2 &= 89^2; \\ (F_4 F_7)^2 + ({}_2 F_5 F_6)^2 &= (F_{12})^2, \end{aligned}$$

that is, the Fibonacci triangle equality. In general, this property may be proved for any four subsequent Fibonacci numbers. Consider $F_n, F_{n+1}, F_{n+2}, F_{n+3}$ then

$$(F_n F_{n+3})^2 + (2F_{n+1} F_{n+2})^2 = F_{2(n+2)}^2 \quad (P1)$$

To prove (P1), we may write from (R1), $F_n F_{n+3} = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = F_{n+2}^2 - F_{n+1}^2$.

From (P1), $(F_{n+2}^2 - F_{n+1}^2)^2 + 4F_{n+1}^2 F_{n+2}^2 = F_{n+2}^4 + 2F_{n+2}^2 F_{n+1}^2 + F_{n+1}^4 = (F_{n+2}^2 + F_{n+1}^2)^2 = F_{2(n+2)}^2$.

As noted by Hagspihl, for a right angled triangle with the two shorter sides as $(F_n F_{n+3})$ and $2(F_{n+1} F_{n+2})$ then the area is equal to $F_n F_{n+1} F_{n+2} F_{n+3}$.

The Fibonacci recurrence relation (R1) has the characteristic equation $x^2 - x - 1 = 0 = (x - \alpha)(x - \beta)$, which produces two roots, where

$$\alpha = \text{the golden ratio} = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} \quad (P1.1)$$

Using these two roots we obtain the classical Binét form $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

The Fibonacci sequence $\{F_n\}$ may also be written in terms of a sum such that

$$F_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n/2}{n/2}, \quad (P3)$$

where $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x , (the floor function). The Fibonacci numbers may also be written as the product of trigonometric functions,

$$F_n = \prod_{j=1}^n \left\{ 1 - 2i \cos\left(\frac{nj}{n+1}\right) \right\}. \quad (P4)$$

One of the most important properties of the Fibonacci numbers, known as Zeckendorf's theorem [8] is the special way in which they can be used to represent integers. Every positive integer has a unique representation of the form $n = F_{r_1} + F_{r_2} + \cdots + F_{r_j}$, where $r_1 \gg r_2 \gg \cdots \gg r_j \gg 0$.

$$\begin{aligned} 1,000,000 &= F_{30} + F_{26} + F_{24} + F_{12} + F_{10} \\ &= 832,040 + 121,393 + 46,368 + 144 + 55. \end{aligned}$$

Relation

A more general relation is the generalized Fibonacci polynomials

$$g_{n+1} - bg_n - cg_{n-1} = 0, \quad g_0 = 1, \quad (P5)$$

where b and c are real numbers, from which we obtain the polynomials (see diagram).

The recurrence relation (P5) is more general than (R1) because for various values of b and c we may obtain the Jacobsthal, Pell, Fermat, Čebyšev and Fibonacci sequences.

The characteristic equation of (P5) is $x^2 - bx - c = 0$ from which we obtain the two roots

$$\xi_0 = \frac{b + \sqrt{b^2 + 4c}}{2} \quad \text{and} \quad \xi_1 = \frac{b - \sqrt{b^2 + 4c}}{2}.$$

n	g_n
0	1
1	b
2	$b^2 + c$
3	$b^3 + 2bc$
4	$b^4 + 3b^2c + c^2$
5	$b^5 + 4b^3c + 3bc^2$
6	$b^6 + 5b^4c + 6b^2c^2 + c^3$
7	$b^7 + 6b^5c + 10b^3c^2 + 4bc^3$
8	$b^8 + 7b^6c + 15b^4c^2 + 10b^2c^3 + c^4$
9	$b^9 + 8b^7c + 21b^5c^2 + 20b^3c^3 + 5bc^4$
10	$b^{10} + 9b^8c + 28b^6c^2 + 35b^4c^3 + 15b^2c^4 + c^5$

The sequence $\{g_n\}$ may then be expressed as

$$g_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} c^r b^{n-2r} = \frac{\xi_0^{n+1} - \xi_1^{n+1}}{\xi_0 - \xi_1} = \frac{1}{\sqrt{b^2 + 4c}} \left[\left(\frac{b + \sqrt{b^2 + 4c}}{2} \right)^{n+1} - \left(\frac{b - \sqrt{b^2 + 4c}}{2} \right)^{n+1} \right].$$

Many other identities of this form may be obtained, for example utilizing (P4) we may conclude that

$$\sum_{r=0}^n \binom{2n-r}{r} \left(-\frac{1}{4} \right)^r = 2^{-2n} (2n+1) = \prod_{j=1}^n \sin^2 \left(\frac{\pi j}{2n+j} \right).$$

Padoran and Perrin

The Padoran integer sequence, see Sloane [5] and Stewart [7], is defined by the recurrence relation

$$P(n) - P(n-2) - P(n-3) = 0; \quad n \geq 3 \tag{P6}$$

with initial condition $P(0) = P(1) = P(2) = 1$. The characteristic equation of (P6) is

$$x^3 - x - 1 = 0 \tag{P7}$$

with roots $\rho_0 = 1.324717957$,

$$\rho_1 = -0.6623589786 + 0.5622795121i \quad \text{and}$$

$$\rho_2 = \bar{\rho}_1$$

where $\bar{\rho}_1$ is the complex conjugate of ρ_1 . The closed form representation of (P6) can be written as

$$P(n) = \sum_{j=0}^2 \frac{1 + p_j}{p_j^{n+2} (2 + 3p_j)}$$

The Perrin integer sequence, see Sloane [5] and Perrin [4], is defined by (P6) but with the initial conditions $P(0)=3$, $P(1)=0$, $P(2)=2$. It has the same characteristic equation (P7) and in this case the closed form representation of the Perrin sequence is $P(n) = \sum_{j=0}^2 p_j^n$. The plastic constant is defined as the limiting ratio of the successive terms, and therefore $\lim_{n \rightarrow \infty} \frac{P(n+1)}{P(n)} = \rho_0$.

Sofa [6] has extended the Padovan and Perrin integer sequences by considering the recurrence relation
$$S(n+1) + 2S(n) - S(n-2) = 0 \tag{P8}$$

with initial conditions $S(0)=1$, $S(1)=-2$ and $S(2)=4$. Hence we obtain the alternating sign terms $\{S(n)\} = \{1, -2, 4, -7, 12, -20, 33, -54, 88, -143, 232, \dots\}$.

The characteristic equation of (P8) is $x^3 + 2x^2 - 1 = 0$ with roots $x_0 = -1$, $x_1 = -\alpha$ and $x_2 = -\beta$, where α and β are the classical Fibonacci numbers (P1.1).

In terms of binomial sums and hypergeometric functions we may write

$$\begin{aligned} S(n) &= \sum_{r=0}^{\lfloor n/3 \rfloor} \binom{n-2r}{r} (-2)^{n-3r} = (-2)^n {}_3F_2 \left[\begin{matrix} -\frac{n}{3}, & \frac{-n+1}{3}, & \frac{-n+2}{3} \\ \frac{-n+1}{2}, & -\frac{n}{2} \end{matrix} \middle| \frac{27}{32} \right] \\ &= (-1)^n \left[-1 + \frac{1}{\sqrt{5}} \left\{ \frac{\beta^{n+2}}{\alpha} - \frac{\alpha^{n+2}}{\beta} \right\} \right]. \end{aligned} \tag{P9}$$

The plastic constant

$$\lim_{n \rightarrow \infty} \left| \frac{S(n+1)}{S(n)} \right| = \alpha, \quad \text{the golden ratio.}$$

An interesting expression of the infinite form of (P9) can be written as, see Sofa [6]

$$S_\infty = \sum_{r=0}^{\infty} \binom{n-2r}{r} (-2)^{n-3r} = (-1)^n \frac{\alpha^{n+1}}{3\alpha - 4}.$$

Not all sequences are necessarily easy to represent as a recurrence relation or in closed form. Consider the sequence

$$\begin{aligned} \{R(n)\} &= \{1, 1, 1, -1, -3, -5, -3, 3, 13, 19, 13, -13, -51, -77, -51, 51, 205, \\ &\quad 307, 205, -205, -819, -1229, -819, \dots\}, \end{aligned}$$

at first sight it looks intractable, in fact, we can represent it by the recurrence relation

$$R(n+1) = R(n) - 2R(n-2) \tag{P10}$$

with initial conditions $R(0) = R(1) = R(2) = 1$. A closer look at (P10) indicates that it is similar to (P8) with the coefficients of $S(n)$ and $S(n-2)$ changed to -1 and 2 respectively. The recurrence (P10) has the characteristic equation $x^3 - x^2 + 2 = 0$ with roots $\{-1, 1+i, 1-i\}$, and can be represented in binomial and hypergeometric form as

$$R(n) := \sum_{r=0}^{\lfloor n/3 \rfloor} (-1)^r \binom{n-2r}{r} 2^r = {}_3F_2 \left[\begin{matrix} -\frac{n}{3}, & \frac{1-n}{3}, & \frac{2-n}{3} \\ -\frac{n}{2}, & \frac{1-n}{2} \end{matrix} \middle| \frac{27}{2} \right] = \frac{1}{5} \left[(-1)^n + 2^{1+\frac{n}{2}} \left\{ 2\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right\} \right].$$

Some other sums and products can be found in an article by Borwein and Corless [2].

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