The CME Project

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Abstract

The *CME Project* is a four-year, National Science Foundation-funded high school program organized around the mathematical themes of algebra, geometry, and analysis. This talk presents and discusses the following core principles of this curriculum:

- fostering mathematical habits of mind;
- separating convention and vocabulary from matters of mathematical substance;
- developing general-purpose mathematical tools;
- learning through experience before formalizing;
- placing high expectations on all students.

In addition, examples will be given of some of the distinguishing features of the *CME Project*. These include design features such as *Getting Started* (a feature that launches each investigation) and dialogues among fictional students. The talk will finally present some examples of specific mathematical habits that are developed in the program, including "abstracting regularity from repeated calculations," a form of encapsulation, and "reasoning by continuity," a useful habit in calculus and analysis.

Introduction

The *CME Project* is a four-year, National Science Foundation-funded high school program organized around the mathematical themes of algebra, geometry, and analysis. The program aims to help students and teachers in the US experience the thrill of solving problems and building theories, and at the same time appreciate the standards of rigor that are central to mathematical culture. The curriculum developers' goal is of building a program around mathematics, as a set of ways of thinking—an evolving set of habits of mind—as well as a body of results that have been derived, over the centuries, through those ways of thinking.

The program is structured as four courses, according to the American tradition: algebra, geometry, advanced algebra, and pre-calculus. Unlike traditional texts driven by low-level skill development, however, the *CME Project* gives teachers the option of a problem-based, student-centered program, and the program is organized around mathematical themes familiar to teachers and parents.

The team of curriculum developers and advisory boards members is truly eclectic, and it is, in itself, one of the distinguishing features of this curriculum. The *CME Project* developers believe criticism from every corner of the mathematical community to be a key to success. Therefore the *CME Project* materials undergo a process of intense criticism from a community of mathematicians, teachers, mathematics educators, cognitive scientists, education researchers, curriculum developers, specialists in educational technology, and teacher educators. During very spirited advisory board meetings and weekly staff meetings different perspectives on the *CME Project* materials are shared and analyzed. Extensive revisions are made to the drafts based on this feedback. This paper describes some core principles around which the *CME Project* is built, followed by a few other distinguishing features of the program.

Fostering Mathematical Habits of Mind

The primary goal of this curriculum is for students to become familiar with mathematical language, and to learn to think and work using mathematical habits of mind. The curriculum's organization mirrors the way ideas are organized in mathematics itself: around the themes of algebra, geometry, and analysis. "The *CME Project* team sees these branches of mathematics not only as compartments for certain kinds of results, but also as descriptors for *methods* and *approaches*—the habits of mind that determine how knowledge is organized and generated within mathematics itself" (Cuoco, 2006a; emphasis in the original). One of the core beliefs of the program is that the widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking used to create the results (Cuoco, 2006a).

For example, in *CMEP Geometry*, one of the lessons in Chapter 5 asks students to draw a circle (possibly with geometry software), choose a point *P* inside it, draw a chord through *P* with endpoints *A* and *B*, and look for invariants while moving *A* along the circle. Students notice that the product (*PA*)(*PB*) remains constant. In a very natural way, this leads to defining such product as the "power of a point." Subsequently, through a *Take it Further* problem, students use the habit of "reasoning by continuity" and (again) "looking for invariants" to discover that it makes sense to consider the case in which the point is not inside the circle. This idea leads to a more general definition, in which the power of any point—inside or outside the circle can be found. Choose a line through *P* that intersects the circle. Then the power of *P* is the product of lengths, *PA* times *PB* where *A* and *B* are the points where that line intersects the circle. Later, in *CMEP Precalculus*, the power of a point is defined again, algebraically, and students learn another way to visualize this invariant. To expand their understanding of this idea students learn to "generalize", another important habit of mind. This process is typical of mathematics as a dynamic ever-expanding field of knowledge built upon previous knowledge. When students become part of this process and come to appreciate it, they are thrilled and experience a feeling of empowerment.

The development of such habits of mind (and others) allows students to overcome what seem to be stubborn misconceptions about mathematics (Cuoco, 2006a), but it is, by nature, a process that requires time and focused attention. This is why each lesson of the *CME Project* ends with three types of sets of exercises: "Check Your Understanding", "On Your Own, and "Maintain Your Skills", that guarantee the students sustained exposure and focused work on the habits of mind they are developing.

Separating Vocabulary from Matters of Mathematical Substance

As students are becoming familiar with the language of mathematics, it is important for them to distinguish vocabulary from matters of substance. The development of an idea requires exploration, observation, conjecture, and formalization, complex and delicate mathematical processes, while the names and symbols that can be (or historically have been) chosen to describe the idea are simply conventions and vocabulary. One way in which the *CME Project* tries to make this distinction clear to students is by using textured emphasis to highlight the separation. To help students understand the difference between definition and consequence, the program contains paragraphs labeled: "Ways to Think About It", "For Discussion", "In-Class Experiment", or "Conjecture" (to mention a few), which contain matters of mathematical substance; and others labeled: "Facts and Notation", "Definition", or "Historical Notes". For example, a paragraph labeled "Facts and Notation" in the *Student Edition* describes how the choice of the positive square root for the radical symbol is convention, which is important, but not inherent in the mathematics itself. Later, the text discusses the merits of the choice for the measure of "badness" of a regression line (sum of squares of differences in y-heights). This is highlighted as a matter of mathematical substance.

Developing General-Purpose Mathematical Tools

As described above, the *CME Project* focuses on mathematical habits of mind, ways of thinking, and tools that lead to the construction of new knowledge built upon the previouslyconstructed knowledge. Unlike what is too often presented to students as mathematics, these tools are not a random collection of results, rules, or techniques that only work locally, but a collage of methods that can be refined to handle constructions that are more and more mathematically sophisticated. General-purpose tools are not constrained to "fit" a certain type of problem; instead they are based on an analysis of the habits of mind used by mathematicians, and they can therefore be extended well beyond the topics in high school mathematics. For example, in elementary algebra, general-purpose tools the curriculum focuses on are: exploiting regularity in calculations in order to construct generic algorithms, associating an algebraic equation with a geometric object, or chunking pieces of an expression together to reduce a calculation to one already understood (Cuoco, 2008).

Another general-purpose tool developed throughout the curriculum is that of proof. Proof is a tool that builds on deductive reasoning and brings closure to a statement, which has been developed from a conjecture; that is, proof ultimately serves to convince oneself and others of the truth of the statement. Throughout the program students use their proofs as a research technique. For example, understanding the proof of the fact that the segments connecting the midpoints of a quadrilateral form a parallelogram lets students see when that parallelogram is a rhombus. As for all mathematical habits of mind, becoming familiar with proof is a process that requires time and practice. This is why the *CME Project* does not isolate proof, describing it only as an *ad hoc* tool for certain kinds of geometry problems, but it makes use of it (and has students use it as a tool) throughout the whole program.

Experience before Formality

Each *CME Project Investigation* starts with a *Getting Started* lesson, which contains problems and experiments that preview the important ideas in the exposition. These problems and experiments are set in simple numerical and geometric contexts so that students can get a "feel for" the kinds of problems they will be thinking about in the *Investigation*. Then in each lesson, when introducing a mathematical concept, students are guided by questions and activities that narrow the focus of the problem and that lead them to every new definition, conjecture, or theorem. This way, students encounter a definition only when they need it to better describe the objects the definition refers to. This process becomes natural and students are frequently are able to think of a definition on their own. Each lesson is then brought to closure through worked-out examples, written dialogues that codify methods, or the formal statement of a theorem that is the culmination of a series of mathematical "discoveries" yielding its proof.

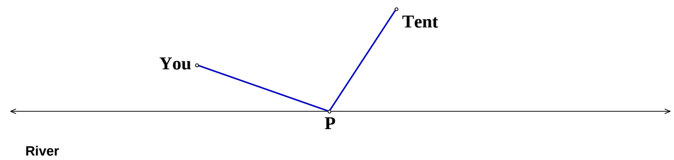
In particular, theorems never "drop out of the sky": activities prepare the students by helping them build their own conjectures and understand the properties underlying the objects they manipulate. This way, when students are ready to state a theorem themselves (and no earlier), the theorem appears in the text, as a milestone that marks new knowledge and that can be used as a stepping-stone from then on. This is the typical way of building new knowledge used by professional mathematicians.

Another Distinguishing Feature: Dialogues

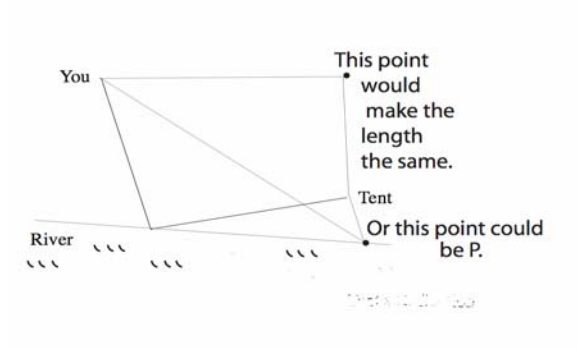
An interesting feature of the *CME Project* is the presence of dialogues among imaginary characters that recur throughout the program. Field-testing has shown that students relate to these characters, who embody typical misconceptions, personalities and ways of thinking. Here is an example of how two characters, Tony and Derman find a new way of solving a problem, referred to in the *CME Project* as the

> **Burning tent problem**: "You are on a camping trip, returning from a walk. Standing at the point marked "You" in the figure below, you notice that your tent (at "Tent") is on fire. You happen to have an empty bucket with you. You need to run to the river, fill your bucket with

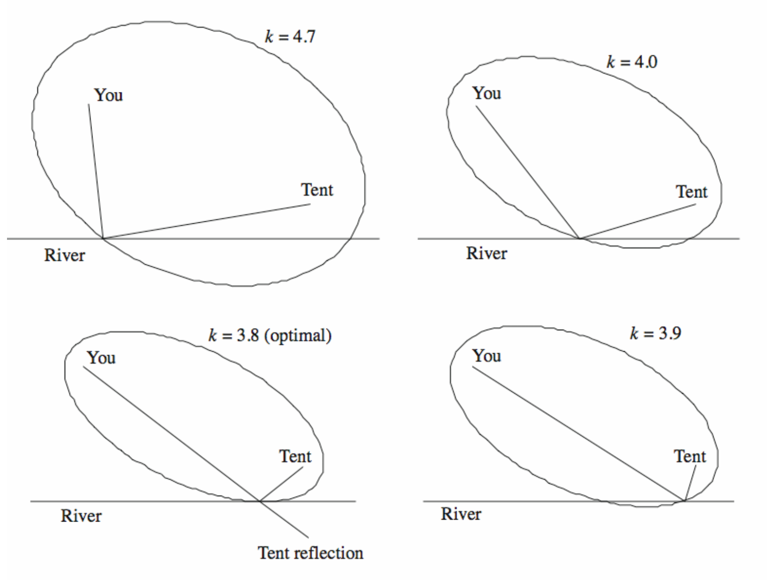
water, and get to the tent. What point on the shore should you stop to fill the bucket to minimize the total distance of the trip?"



Tony: Hey Derman, do you remember the "burning tent" problem? It said...
Derman: Yes, I remember that we tried to find the best spot to fill the bucket in the river and minimize the total distance you had to run.
Tony: Well, I was thinking about solving it by looking at some "wrong" answers like this one.



Derman: Why would you do *that*?
Tony: Surely there are other spots *P* for which the length of the path you run is exactly the same. Look at my picture. If we found a way of getting all the points that are "equally bad" for every choice of a point on the riverbank, we could keep trying points until we found the set of points that are "equally good."



Derman: I don't get it.
Tony: Start by looking for those spots (not along the river) that are just as bad (an equal total distance from You and the Tent) as some particular point along the river. That is, what is the set of points some constant *total* distance from You and the Tent?

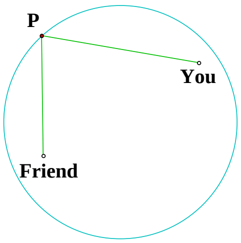
Later, students discover that when this ellipse is tangent to the shoreline, the point of tangency is the optimal point that minimizes the distance.

Low Threshold, High Ceiling

The *CME Project* staff believes that the expectations of mathematics understanding for students and teachers should be maintained at a high level. Many curricula hold low mathematical expectations for students, but our field tests have shown that students at all levels are able to think in a mathematical way and successfully work through the lessons of the *CME Project*. In addition, the *CME Project* design employs a low threshold-high ceiling approach, in that "each chapter starts with activities that are accessible to all students and ends with problems that will challenge the most advanced students" (Cuoco, 2006a). Since the program is so mathematically rich, in each lesson teachers, aided by the *Teacher Edition*, have the opportunity to choose from different topics.

Consider, for example, the burning tent problem discussed in the previous section. It provides a "jumping off" point for teachers and, at the same time, the opportunity for students to pursue rigorous content. Here is one of the problems it can lead to:

"You are in a circular swimming pool and you want to swim to the edge of the pool to drop off your sunglasses, and then swim to your friend. Where should you land on the edge of the pool to keep the trip to a minimum?"



In this chapter on geometric optimization, students have discovered how contour lines and some geometric properties of the ellipse may be useful when solving geometric optimization problems (see the "burning tent" problem above). In particular, the "swimming pool" problem can be solved by drawing a contour plot that represents sets of points for which the sum of the distances to *P* from "You" and "Friend" (the foci) is constant. These contour lines are ellipses, and the trip will be minimum in the point(s) of tangency of the "smallest" ellipse with the circle (here "smallest" means the ellipse that has at least one point of tangency with the circle and that represents the set of points with the smallest sum of distances from the foci).

To students who still find this problem not challenging enough, the teacher is encouraged to raise it to yet a higher level by asking other questions, such as: "If the point "You" is fixed, but "Friend" is allowed to move in the pool, when would the problem have two solutions? When would it have one solution? When would it have no solutions? What geometric/analytical interpretation can you give?" Therefore, the type of question, the choice of tools students can use to reason about the question, and the level of precision of their answers are left to the teacher's discretion.

Another Habit of Mind

Another habit of mind that is developed throughout the program is "abstracting regularity from repeated calculations," a form of encapsulation. This is a very important mathematical habit of mind, since many highly theoretical investigations in algebra begin with a careful analysis of patterns that emerge from repeated calculations, leading to functions defined by algorithms (Cuoco, 2008). For example, thanks to this habit of mind, students learn to model situations with equations. Here is a problem students in an algebra class typically struggle with:

"Mary drives from Boston to Washington, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes 18 2/3 hours, how far is Boston from Washington?"

On the other hand, pre-algebra students usually have no difficulty solving the following:

"Mary drives from Boston to Washington, a trip of 500 miles. If she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back, how many hours does her trip take?"

The CME Project proposes an effective method, referred to as the *Guess-Check-Generalize* tool, which essentially exploits students' ability to solve a pre-algebra problem to help them construct "the equation" whose solution settles the algebra problem (Cuoco, 2008). The first step is to guess at an answer: not stumble on the right answer, rather to focus on the steps one takes to check the guess. This leads to something like:

"You take the guess, divide it by 60, then divide it by 50, add your answers and see if you get $18\frac{2}{3}$."

which means: $\frac{\text{guess}}{60}+\frac{\text{guess}}{50}\stackrel{?}{=}18\frac{2}{3}$ or $\frac{x}{60}+\frac{x}{50}\stackrel{?}{=}18\frac{2}{3}$

which leads to the "pure" algebra solution. In brief, this tool allows you to carry out several concrete examples of a process that you don't quite "have in your head" in order to find regularity and to build a generic algorithm that describes every instance of the calculation. Therefore the *Guess-Check-Generalize* tool not only captures a very common habit that is useful throughout algebra, but it is also extensible: the encapsulation spiral is infinite in both directions so that the algorithms can eventually be encapsulated into objects in their own right and passed off to "higher-order" operators like function composition, the derivative, and the difference operator (Cuoco 1995).

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