Reasoning Algebraically about Operations: Developing early algebraic thinking by examining the generalizations that underlie young student's mathematical thinking. What do teachers and those who prepare teachers need to understand? (Workshop)

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Abstract
For the last decade, we have directed projects with teacher-collaborators to investigate the ideas of "early algebra." Our data consist of cases both print and video intended to capture students' thinking as it finds expression in classroom process. We have found that as children learn about the four basic operations—understanding the kinds of situations the operations can model, sorting out various means of representing them, and figuring out how to compute efficiently—they observe and comment upon regularities in the number system. For example, they may notice that the calculations 72-38 and 74-40 produce the same result, or that successive answers to a series of problems (10+1=?, 10+2=?, 10+3=?, …) increase by 1. Such regularities, emerging naturally from children's work in arithmetic, become the foundation not only for exploration of generalizations about number and operations, but also of the practices of formulating, testing, and proving such generalizations—and it is these practices that are at the heart of what we mean by "early algebra." In this interactive session, we will examine video and print cases to discuss our findings and the implications of this work for teachers, for those who prepare prospective teachers and for those who work with in-service teachers.

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Workshop Summary: We (Deborah Schifter, Susan Jo Russell, and Virginia Bastable) have been investigating the kinds of generalizations children make in the context of their work in arithmetic and what it means to help teachers recognize those generalizations in their student's work. The students (ages 5 through 12) use verbal descriptions and visual representations such as modeling with cubes or creating drawings to express their general statements. In this session we will look at video and print cases to analyze the thinking of the children, to examine what generalizations are they expressing or acting upon, how the teachers' questions and moves might support such thinking, and how we ourselves would express those generalizations in symbolic notation. Finally we will consider what is it that teachers need to learn in order to support such discussions.

This presentation draws from video and print cases in *Reasoning Algebraically about Operations (RAO),* a module of the Developing Mathematical Ideas (DMI) Professional Development Series. This excerpt from the introduction to the RAO casebook [1]sets the stage:
For most adults, the use of letters to stand for numbers—x's and y's—is the chief identifying feature of algebra. However, underlying such notation are ways of reasoning about how the

[1] Schifter, D., Bastable, V., and Russell, S.J. (with S. Monk). (2008). *Reasoning Algebraically about Operations, Casebook*. Parsipanny, NJ: Dale Seymour Publications, Pearson Learning Group.

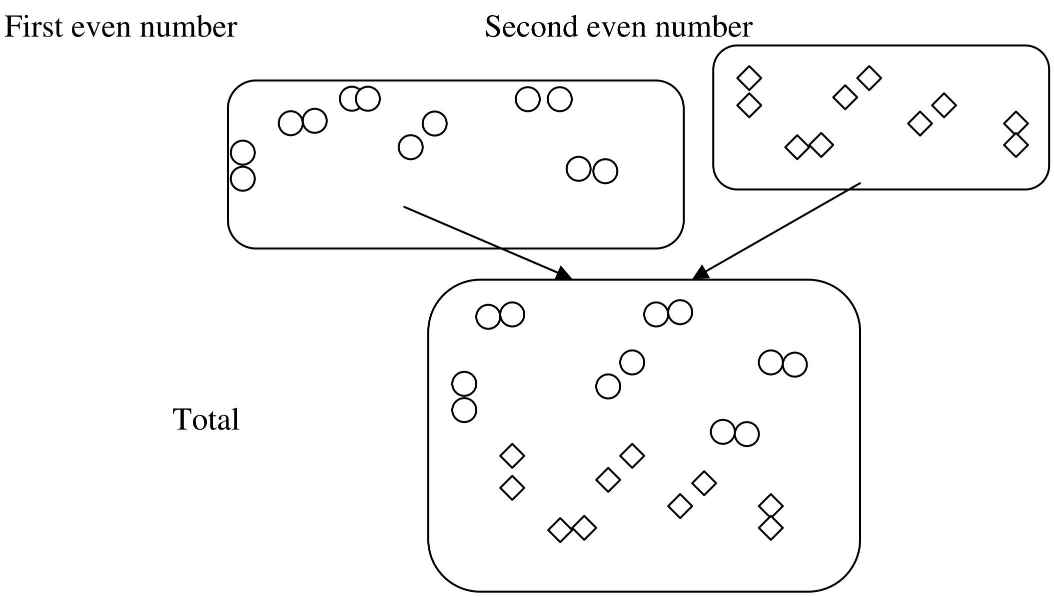
operations work. This *reasoning* about operations, rather than the notation, is the work of elementary students in algebra. In *Reasoning Algebraically about Operations*, we examine the *generalizations* students make about the operations and the reasoning entailed in addressing the question, Does this always work?

For example, consider the following vignette:

An elementary-grade class was exploring odd and even numbers and several children noticed that whenever they added two even numbers, the result was an even number. Based on this evidence, several of the children were ready to declare that whenever you add any two even numbers, you have to get an even number. Others acknowledged that each time they tested it, it came out that way, but you can't tell if it will always happen because "numbers go on forever."

Myra said she knows it will happen for all even numbers. Because, she said, take two even numbers. You can make them into pairs with none left over. Then, when you put the numbers together, you combine the parts so that your new number has pairs with none left over.

Myra illustrated what she meant with cubes and disks:



Felicity worked hard to understand what Myra had said. But as she began to reiterate it, she went back to the idea that you can't know for all numbers. "There might be two even numbers somewhere that, when you add them together, make an odd. You just don't know."

Myra insisted she did know, for all even numbers.

In this vignette we see that as students move from particular numbers and actions to *patterns of results*, they begin to make generalizations. That is, they begin to view numbers and operations *as a system.* But the vignette also raises several questions: What does it mean to make a generalization—a claim that something is *always* true? What does it mean to prove a generalization when you are making a claim about an infinite class of numbers and so cannot check every case? How do children engage with these questions and what constitutes “proof” at this level? When students begin to state and justify their own generalizations, they tend to use diagrams, concrete objects, and words to do so—just as Myra did in her statement about adding two even numbers. As their statements become more complicated, they begin to need other ways to point at “the first number,” or “the bigger number,” “the answer you get when you add two numbers,” and so forth. This is the beginning of what later becomes conventional algebraic notation.

In this presentation we will examine the thinking of students (of ages 5 – 10) as they engage in making, testing, and proving generalizations they have noted in the context of their work on arithmetic. Through two video and one print case we will examine these questions:
What generalization(s) do first graders express as they work to develop a computational strategy for addition?
What generalization does a second grade class articulate as they examine a set of related addition and subtraction problems?
What generalization does a fourth grade class articulate as they examine a pair of related subtraction problems?

Additionally, we will address a related set of questions: What is it that teachers need to understand in order to recognize, develop and support this kind of mathematical reasoning in their students and how might they learn it?

Our work has been incorporated into the professional development series, Developing Mathematical Ideas (DMI). DMI is a professional development curriculum designed to help teachers think through the major ideas of K-7 mathematics and examine how children develop those ideas. At the heart of the materials are sets of classroom episodes (cases) illustrating student thinking as described by their teachers. In addition to case discussions, DMI seminar sessions offer teachers opportunities: to explore mathematics in lessons led by facilitators; to share and discuss the work of their own students; to view and discuss videotapes of mathematics classrooms; to write their own classroom episodes; to analyze lessons taken from innovative elementary mathematics curricula; and to read overviews of related research.

Each DMI modules supports eight three-hour sessions of professional development work for teachers. A DMI module consists of a casebook, a facilitator’s guide, and a DVD. There are seven DMI[2] modules, each focused on a particular mathematical strand:

Building a System of Tens Participants explore the base-ten structure of the number system, consider how that structure is exploited in multi-digit computational procedures, and examine how basic concepts of whole numbers reappear when working with decimals.

Making Meaning for Operations Participants examine the actions and situations modeled by the four basic operations. The seminar begins with a view of young children's counting strategies as they encounter word problems, moves to an examination of the four basic operations on whole numbers, and revisits the operations in the context of rational numbers.

[2] This website has more information on the DMI materials and programs: http://www2.edc.org/CDT/dmi/dmicur.html

Geometry: Examining Features of Shape Participants examine aspects of 2D and 3D shapes, develop geometric vocabulary, and explore both definitions and properties of geometric objects. The seminar includes a study of angle, similarity, congruence, and the relationships between 3D objects and their 2D representations.

Geometry: Measuring Space in One, Two and Three Dimensions Participants examine different attributes of size, develop facility in composing and decomposing shapes, and apply these skills to make sense of formulas for area and volume. They also explore conceptual issues of length, area, and volume, as well as their complex inter-relationships.

Working with Data Participants work with the collection, representation, description, and interpretation of data. They learn what various graphs and statistical measures show about features of the data, study how to summarize data when comparing groups, and consider whether the data provide insight into the questions that led to data collection.

Reasoning Algebraically about Operations Participants examines generalizations at the heart of the study of operations in the elementary grades. They express these generalizations in common language and in algebraic notation, develop arguments based on representations of the operations, study what it means to prove a generalization, and extend their generalizations and arguments when the domain under consideration expands from whole numbers to integers.

Patterns, Functions, and Change Participants discover how the study of repeating patterns and number sequences can lead to ideas of functions, learn how to read tables and graphs to interpret phenomena of change, and use algebraic notation to write function rules. With a particular emphasis on linear functions, participants also explore quadratic and exponential functions and examine how various features of a function are seen in graphs, tables, or rules.