# Developing Algebraic Habits of Mind from the Context of Computer Science

## Workshop: Applications of Mathematics and Modeling in the Real World

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**Abstract**
The workshop will highlight hands-on activities used in the professional development project that illustrate each of the three algebraic habits of mind, as stated by Mark Driscoll (*Fostering Algebraic Thinking*, Heinemann, 1999). These activities will relate mathematics to some aspect of technology. Each algebraic habit of mind will begin with a problem, require generalization of the solution that demonstrates an algebraic habit of mind, and will have an application to the discipline of computer science. The themes of the activities are as follows:

- Abstracting from Computation-Generalization from Computing in Base Two
- Building Rules to Represent Functions-Generalization from Recursive Thinking
- Doing and Undoing-Problem Solving Techniques Requiring Backtracking

The presenters will also give a description of the success of the activities and their transfer to the middle school classroom.

## Introduction

A recent report from the organization for Economic Cooperation and Development reported that young Americans may not be adequately prepared to continue Western Economic dominance. Developing countries are now in the position to keep their citizens who are technologically-savvy at home instead of exporting them to the United States. The United States will no longer be in the position of filling employment positions related to technology with citizens of other countries. Thus, the need arises for American children to experience applications of mathematics that promote interest in technological careers.

## Assumptions

Based on the assumption that the thinking required in computer science is most closely aligned to the type of thinking used in mathematics, mathematics lessons emphasizing the connection between mathematics and computer science seem a natural way to introduce middle school students to applications in mathematics. While conducting a professional development workshop involving middle school mathematics teachers, we found teachers seldom relate mathematics to students' lives and/or a variety of careers and professions. In this regard, teachers of these students lack the background knowledge for implementation of such lessons in their classrooms. As a result, a professional development project for middle school mathematics teachers was designed that focused on mathematics activities emphasizing the connection between mathematics and computer science.

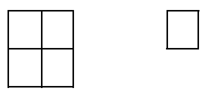
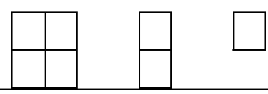
## Habits of Mind

### Abstracting from Computation-Generalization from Computing in Base Two

### ACME Potato Chip

The ACME Potato Chip Company received six freight cars supposedly full of potatoes in 100-lb bags. It was learned that the automatic weighing machine was broken for a while and that some of the freight cars were full of 90-lb bags of potatoes. Ms. Jones said, "Let's load some sacks of potatoes from each freight car into our truck. Then, in one weighing, we will locate the freight cars containing sacks with the 90-lb. bags. Let's take 1 bag of potatoes from the first freight car, 2 from the second, 4 from the third, 8 from the fourth, 16 from the fifth and 32 from the sixth. So, we will have a total of 63 bags. If all the sacks weigh 100 lb, the correct answer from the weighing should be 6300 lb more than the truck. Suppose the answer is 5870 lb more than the truck (i.e., the potatoes in the truck are 430 lb too light). Because each sack differs from 100 lb by 10 lb, there are 43 light sacks in the truck."

**Question to consider**
How could Ms. Jones determine in one weighing which freight cars had bags of potatoes weighing 90 lb instead of 100 lb? Generalize this procedure.
After working with the potato chip problem and finding solutions a variety of ways, we find that by generalizing this situation in base two, we are able to see connections between solutions and the expanded notation in base two. An exercise for connecting base two with base 10 follows.

| **Number** | **Model of Number** | **Expanded Notation** | **Base 10 Representation** |
|---|---|---|---|
| $101_2$ |  | $1(2)^2 + 0(2)^1 + 1(2)^0$ | $4 + 1 = 5$ |
| $111_2$ |  | $1(2)^2 + 1(2)^1 + 1(2)^0$ | $4 + 2 + 1 = 7$ |

The number 43 can be written as $101011_2$ which can be expanded to $1(2)^5 + 0(2)^4 + 1(2)^3 + 0(2)^2 + 1(2)^1 + 1(2)^0$. In the potato chip problem, 0 and 1 represent a type of true-false situation: 0 meaning it is "false that an underweight bag of potatoes is in the car" and 1 meaning it is "true that an underweight bag of potatoes is in the car." Adjusting the initial condition for the powers of 2 in the expanded notation leads to a solution that cars 2, 4 and 6 contain underweight bags of potatoes. A table is provided to help in the generalization for any 3-digit number *ABC* in base 2: $A(2)^2 + B(2)^1 + C(2)^0$. Further generalization of an *n*-digit base 2 number leads to $d_{n-1}(2)^{n-1} + \cdots + d_2(2)^2 + d_1(2)^1 + d_0(2)^0$, where each $d_i$ is either 0 or 1 for $1 \le i \le n$. The representation of $d_{n-1} \ldots d_2 d_1 d_0$ (base 2) can be linked to how computers communicate with one another.

**Building Rules to Represent Functions-Generalization from Recursive Thinking**
**The Martian Invasion**

This activity begins with a simulation of a Martian invasion in which one person in the audience sees a picture of a "Martian." This represents the initial sighting during a Martian invasion. One year later, the initial observer describes the creature to one other person. Every year each person tells only one other person about the Martian. This process repeats until everyone in the audience has heard a description of the Martian.
In 1938, Orson Welles interrupted a program on the radio to dramatize an invasion by Martians. Numerous listeners panicked. Switchboards jammed in police stations and newspapers; drivers sped out of New York; sailors were ordered back to their ships; people saw Martians.

Pretend that the creature we saw was actually a Martian that invaded the state of Texas. As you know, Texans never panic! Instead, they are selective about who they tell; they only tell other Texans. If each person tells only one person about the Martian invasion every year and every person that hears about the invasion, tells only one person about the Martian invasion every year, how many years would it take everyone in Texas to hear about the invasion?

Problem Solving Strategies related to Building Rules to Represent Functions

- Organize information in a Table
- Start Simple
- Look for a Pattern

To solve the problem, students will need to know the population of Texas. In 2000, the population was 20,851,820 people with a projected population of 31,830,589 in the year 2030 (projection provided by Population Estimates and Projections Program, Office of the State Demographer, http://txsdc.utsa.edu/tpepp/2006projections/2006_txpopprj_txtotnum.php)
**Question to ask:** How can you find the number of people who know about the Martians in any year? *To get the number of people who know the next year, multiply the number of people who know the previous year by 2.*
Response indicates an Implicit Rule or Recursive Rule of Next = 2 * Now, Start = 1. This type of response connects directly to computational methods used by technology and lends themselves to solution methods requiring the use of technology.

**Investigating the Problem Using Calculators**
As information is organized in a table, a pattern of doubling is identified. A calculator may be used to compute values and to solve the problem. Setting the constant key on the TI-73 at 2 assists in the solution method. Many students will start with 1 because initially one person knows. The pattern displayed on the homescreen of the calculator indicates that the TI-73 interprets this as an initial condition, i.e. initially in year $n = 0$, one person knows. Therefore, $n = 1$ means in year 1, 2 people know, that is, year 1 is the first year someone was told about the Martian.
Solution: It will take about 25 years for everyone in Texas to know.

**Investigating the Problem using a Spreadsheet**
Recursive thinking is necessary for setting up a spreadsheet. Variable notation is introduced through expressing each cell's entry by a row number and column letter resulting in a Next-Now rule, starting with an initial value. The population of Texas in 2000 and its projected population in 2030 serve as a means to compare the population amounts to the number of people who know each year. The repeated multiplication by 2 can be seen in the formulas which after entry into the cells appear.

## Doing and Undoing-Problem Solving Techniques Requiring Backtracking
## Cars and Motorcycles

All 20 parking spaces in my favorite parking lot are filled. Some are occupied by motorcycles and others by cars. Some people count to 10 when they get angry, but that wasn't nearly far enough. I counted wheels--66 to be exact. How many cars and how many motorcycles have invaded my territory?

Solve the problem using at least two different methods. Identify any algebraic thinking or algebraic habits of mind applied in your methods. Justify your answer.

**Solutions**

| **Number of Cars** | **Number of Motorcycles** | **Number of Car Wheels** | **Number of Motorcycle Wheels** | **Total Number of Wheels** |
|---|---|---|---|---|
| 20 | 20 - 0 | 4(20) | 2(20 – 20) | 4(20) + 2(20 –20) = 80 |
| 19 | 20 - 19 | 4(19) | 2(20 – 19) | 4(19) + 2(20 – 19) = 78 |
| 18 | 20 - 18 | 4(18) | 2(20 – 18) | 4(18) + 2(20 – 18) = 76 |
| 17 | 20 - 17 | 4(17) | 2(20 – 17) | 4(17) + 2(20 – 17) = 74 |
| ⋮ | | | | |
| $C$ | 20 - $c$ | 4($c$) | 2(20 – $c$) | 4($c$) + 2(20 – $c$) = 66 |

The table above depicts the process of solving the problem through Guess and Test. Through the process, one "does" calculations to "undo" the equation which will give the answer to the problem. The table shows how the process can be generalized by "Building a Function Rule": $y = 4(c) + 2(20 - c)$. What is $c$ when $y = 66$? Thus, the function was evaluated at some point to produce 66 and now must be solved to find $c$ (do and undo).

In word processing, spreadsheets, and databases, "do and undo" is present through commands that seem rather trivial to the technology user. These do-undo commands must be programmed. However, the idea of reversibility which depicts a type of do-undo is present in some programming languages, such as Prolog. The program CARCYCLE3 written in this programming language will find the solution to the Cars and Motorcycle problem. This language illustrates the use of reversibility (backtracking) as it searches to "undo" the equation $66 = 4(c) + 2(20 - c)$.

Determine where "do and undo" in terms of reversibility is used by Prolog. Note variables or assignment statements and what they represent. Compare the program's "thinking" to the thinking used in solving the Cars and Motorcycles problem by you or your students.

CARCYCLE3:

```
num(X) :- X = 20; X=19; X=18; X=17; X=16; X=15; X=14; X=13; X=12; X=11; X=10; X=9;
X=8; X=7; X=6; X=5; X=4; X=3; X=2; X=1; X=0.
park(Car, Cycle):- num(Car), Cycle is 20 - Car, write('Trying: Cars = ' ), write(Car), write('
Cycles= '), write(Cycle), nl, TotalWheels is Car * 4 + Cycle * 2, TotalWheels = 66.
| ?- park(Car, Cycle).
Trying: Cars = 20  Cycles= 0
Trying: Cars = 19  Cycles= 1
Trying: Cars = 18  Cycles= 2
Trying: Cars = 17  Cycles= 3
Trying: Cars = 16  Cycles= 4
Trying: Cars = 15  Cycles= 5
Trying: Cars = 14  Cycles= 6
Trying: Cars = 13  Cycles= 7
Car = 13
Cycle = 7 ? ;
Trying: Cars = 12  Cycles= 8
Trying: Cars = 11  Cycles= 9
Trying: Cars = 10  Cycles= 10
```

Trying: Cars = 9 Cycles= 11
Trying: Cars = 8 Cycles= 12
Trying: Cars = 7 Cycles= 13
Trying: Cars = 6 Cycles= 14
Trying: Cars = 5 Cycles= 15
Trying: Cars = 4 Cycles= 16
Trying: Cars = 3 Cycles= 17
Trying: Cars = 2 Cycles= 18
Trying: Cars = 1 Cycles= 19
Trying: Cars = 0 Cycles= 20
(15 ms) no
| ?-

In the program num is assigned the variable *X* and has the possible values for *X* to start with 20 and to decrease to 0 by increments of 1. The next line park(Car, Cycle) assigns Car to the variable *X* and states how to calculate the number of motorcycles. The write commands instruct the program to printout the value of Car and Cycle, to print on a new line, to calculate the total number of wheels, and to check the computed total number of wheels against the needed number of 66. Thus, Prolog is "undoing" the equation in a manner similar to guess and check. It "does" calculations and checks these calculations against needed number of wheels (66) in an effort to "undo" the equation.

## Conclusion

Throughout the workshop hands-on professional development activities emphasized the three algebraic habits of mind. Mathematics lessons emphasizing the connections between mathematics and computer science were explored. In the ACME potato chip problem, we found middle school teachers participating in the project had little or no background for changing base 10 numbers to other bases. In the Martian problem, middle school teachers were unfamiliar with setting the problem up in the calculator to see the progression in the exponential sequence, but had few problems understanding what was happening with the function. In the Car and Motorcycle problem, when participants were asked to compare their algorithm with the programming language, Prolog, they had no problems comparing and understanding their own algorithm with the thinking involved in the undoing (backtracking) found in the programming language. One of our goals was to expand the teachers' background so they could relate mathematics to their students' lives and a variety of careers and professions. Christi, a participant in the professional development workshop stated that "Students need to see how mathematics in your class can impact their future. By showing them a computer program you are drawing in students who are considering a career in technology. It makes your class more relevant in their [the student's] high tech world."

## References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