Relational Notation & Mapping Structures: A Data Analysis Framework

Günhan Caglayan
The University of Georgia sezen@uga.edu

Abstract
Three main type of Representational Unit Coordination (RUC) arose from various usages by middle and high school preservice teachers (PST) when making growing rectangles for special numbers such as prime and composite numbers, summation expressions for odd, even, and counting numbers as well as polynomials in x and y using magnetic color cubes and tiles. In a Multiplicative type RUC, I used a relational notation of the form (a,b) where a and b stand for the corresponding linear units of the growing rectangle represented by the dimension tiles. I also observed more than one additive type RUC's which can be described using a functional notation $\sum f(i)=g(n)$ where the areal quantities f(i)'s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends). There is one more type RUC, in between additive and multiplicative which I named Pseudo Multiplicative type RUC. This occurred for the "Area of the boxes of the same color as a product" in dealing with polynomial rectangles made of color tiles. Moreover, a mapping structure of multiplication and addition operations was spelled out by these PST.

Introduction

Mathematics gives rise to quantities which can be represented with manipulatives. When it comes to represent these quantities, both students and their teacher should be proficient in identifying the characteristics of those quantities. Whole numbers can be expressed in terms of units of 1. For instance, the number 3 can be thought of as the collection of three singleton units. Moreover, in a representational situation, three little black square tiles can be used to model the representational quantity "3." Odd integers can be represented as symmetric L-shaped figures, while even integers can be represented as rectangles with dimensions 2 by half the integer made of color tiles (Caglayan, 2006). Prime and composite numbers have various tiled rectangular representations as well (Caglayan, 2007).

Modeling more complicated expressions such as 2x+y+3 by using color tiles is not as obvious. In the example of 2x+y+3, the term 2x is a collection of two units of x (two purple bars with the model), the term y is 1 unit of y (1 blue bar with the model), and the term 3 is a collection of three units of 1 (three little black squares with the model). Therefore, the expression 2x+y+3 is a collection of a collection of the individual irreducible representational units. One not only has to individually identify each representational unit (one purple bar for the x, one blue bar for the y, and one little black square for the 1), but one has to reconcile a collection of a collection of these irreducible representational units in order to demonstrate that 2x+y+3 can not be simplified any further because 2x, y, and, 3 are unlike terms (representational quantities).

Theoretical Framework

Representation of irreducible quantities as well as bigger ones made of these quantities is reminiscent of the unitizing process. All the little pieces (e.g., each color cube denoting a "1" of a special number, each different size tile piece denoting a "1", an "x", a "y", an "x^2", an "xy", or a "y^2") and their various combinations (e.g., a 4 by 2 rectangle – made of 8 irreducible units of 1 – conceptualized as the unitizing of the even number 8, a 2x+y+3 by x+1 rectangle – made of 2 irreducible units of x^2, 5 irreducible units of x, 3 irreducible units of 1, 1 irreducible unit of y, 1 irreducible unit of xy – conceptualized as the unitizing of the polynomial expression $2x^2+1xy+5x+1y+3$) serve for an essential theoretical construct which I define as Representational Unit Coordination.

In its true nature, coordination is about making various different things work effectively as a whole. In the context of my study, it refers to the conception of unit structures in relation to smaller embedded units within these unit structures, or, bigger units formed via iteration of these unit structures. In the multiplicative situation, for instance, the conception of 5 as 5 units of 1 is one way of coordinating units: 5 as a (composite) unit of 1. As another example, 35 can be coordinated multiplicatively as 5 (composite) units of 7 (composite) units of 1. Unit coordination has been previously studied by various researchers in the mathematics education field. Steffe (1988) for instance, analyzed the coordination of different levels of units in whole number multiplication problems which is reminiscent of a key concept in multiplication, i.e., the notion of composite units. The essence of multiplication lies in fact in distributive rather than repeated additive aspect (Confrey & Lachance, 2000; Steffe, 1992). In the example above, the multiplication of 5 by 7 can be thought as the injection of units of 7 (each being units of 1) into the 5 slots of 5, each slot representing a 1. In this example, the conceptualization of each singleton unit describing a unity, i.e., 1, stands for a first level of unit coordination. Moreover, 5 and 7 can be conceptualized (as composite units of 1) as 5×1 and 7×1, respectively, as a second level of unit coordination. The product 5×7 which denotes 5 (composite) units of 7 (composite) units of 1, can be conceptualized as a third level of unit coordination. Some other researchers also studied unit coordination in a fractional situation (e.g. Lamon, 1994; Olive, 1999; Olive & Steffe, 2002). Work on intensive (e.g., miles per hour) and extensive quantities (e.g., number of hours) reflect unit coordination as well (Schwartz, 1988; Kaput, Schwartz, & Poholsky, 1985).

Representational Unit Coordination (RUC) can be defined as the different ways of categorizing units arising from the modeling of identities on representational quantities as the area as a product and area as a sum of the corresponding special rectangles made of color cubes or tiles. In its most basic sense, area of, e.g., a rectangle, is defined as the product of its two dimensions. The identities on special numbers PST analyzed via color cubes and tiles were always about a rectangle – prime rectangle, composite rectangle, summation of counting numbers, odd and even integers generated as a growing rectangle, and polynomial rectangle. Coordination of a particular rectangle's two dimensions, i.e., the arrangement of these two linear units in a particular order as an ordered pair such as (a, b) or (b, a), defines the first part of my construct RUC which is of multiplicative nature. The analysis of the other important concept, area as a sum (of a special number rectangle), is prone to many more, not necessarily hierarchical, additive type RUC for which the addends, namely the areal units, are expressed as n-tuples in square brackets. RUC has more of a relational aspect, that way.

Context and Methodology

I conducted my study with PST enrolled in the Mathematics Education Program in a university in the southeastern United States. I interviewed 5 PST individually twice during January & February 2007. Duration of each session was about 60-75 minutes and each interview session was videotaped. The focus was on problems on identities for prime and composite numbers along with summation of counting numbers, odd and even integers as well as products and factors of polynomials modeled with magnetic color cubes and tiles. I selected my participants from two different undergraduate level mathematics education classes. Ben, Stacy and John came from the "Concepts in Secondary School Mathematics" class of 11 enrolled preservice teachers while Nicole and Ron came from the "Teaching Geometry and Measurement in the Middle School" class of 22 enrolled preservice teachers. All these five students volunteered to participate in my study. All proper names in this study are pseudonyms.

I used thematic analysis (Boyatzis, 1998) supported by constant comparison of the interviews and retrospective analysis. I also simplified and extended the generalized notation for mathematics of a quantity (Behr et al., 1994) in such a way as to cover identities that equate summation and product expressions of special numbers.

Results

Multiplicative type RUC arose from various usages by the PST such as
"It [areal 6] is [linear] 6 and [linear] 1",
"This [linear x] and this [linear 1] to find this [areal x]",
"When you put this length [linear 1] and that length [linear 1] together",
"This edge [linear 1] right here and this edge [linear 1] right here",
"This edge [linear x] by this [linear x] edge".
For all such usages, I used a relational notation of the form (a,b) where a and b stand for the corresponding linear quantities represented by the dimension tiles. Moreover, a mapping structure of multiplication operation was spelled out by these PST. For instance, "And this area right here is x times y... to find that specific spot", can be represented using a functional notation such as $f:(x,y)\rightarrow xy$. Here, f denotes the multiplication operation which maps the ordered pair of linear units x and y into the corresponding "spot", namely the areal xy.

I observed more than one additive type RUC's which can be described using a functional notation $\sum f(i)=g(n)$ where areal quantities f(i)'s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends).
1- Equal Addends: These are the addends describing a composite number rectangle. With the functional notation, $f(i)=c$, for all i. All PST produced this type.
2- Unit Addends (Type I): PST used these addends mostly when dealing with prime number rectangles. This is a special case for equal addends, $f(i)=c$, for all i, with $c=1$.
3- Symmetric Addends: This type came from Ben's work on the summation of odd integers activity. Ben used these addends to describe the odd integers as symmetric L-shapes. For each symmetric L-shape, there are three addends only, i.e., $n=3$. One of these addends is equal to 1, and the remaining two addends are equal to each other. In other words, with the functional notation, one can write, $f(2)=1$, $f(1)=f(3)$. For example, for the case of areal 9, which denotes an odd integer, $f(2)=1$, $f(1)=f(3)=4$. Note that $f(1)+f(2)+f(3)=4+1+4=9$, i.e., the odd integer itself.
Here is an excerpt from the interview with Ben validating "Symmetric Addends" type RUC:

> Ben: They are all odd [See Figure 1]. And you have this diagonal that goes across [pointing to the main diagonal of the growing square] and you have this same number this way [pointing to the left] and this way [pointing down].
> Interviewer: Tell me more about that, in particular for the white one [the symmetric L-shape representing 7].
> Ben: The white one? OK... This has this 1 on the diagonal, and it has 3 down, and 3 across. It's the same [meaning the same "3"]. And that's being odd...

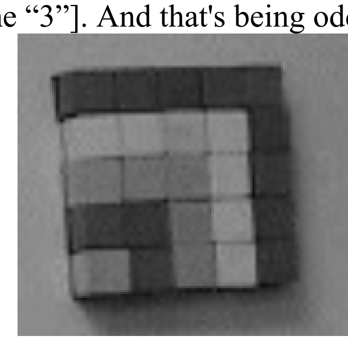

Figure 1. Ben's Work on the Summation of Odd Integers

4- N+(N-1) type Addends: These are the addends describing a symmetric L-shape odd integer. In this case, $n=2$. With the functional notation, $f(1)=N$, and $f(2)=N-1$, i.e., the addends differ only by 1. John, Stacy, and Ben explained their ideas using this type when working on the summation of odd integers activity.

5- (N+1)+(N-1) type Addends: These are the addends describing a nonsymmetric L-shape even integer. Once again n=2. Only John referred to this type when working on the summation of even integers activity. A nonsymmetric L-shape even integer can be described using the functional notation f(1)=N+1, f(2)=N-1.

6- Recursive Addends: f(i+1) is being added to the previous summation (Nicole). With the functional notation, this can be written as g(n+1)=g(n)+f(i+1).

7- Summed Addends: The addends of the growing rectangle are areal units with different shapes made of color cubes representing the "area as a sum" part of the summation formula. For example, f(i)=i, f(i)=2i-1, f(i)=i+(i-1), f(i)=2i, for all i, for the addends corresponding to summation of counting numbers, odd numbers, odd numbers, and even numbers, respectively. 3 out of 5 PST (Nicole, Stacy, John) came up with this usage. Ron and Ben, on the other hand, did not care about the color shapes generating the growing rectangle. Instead, they used "Equal Addends" type RUC in expressing the area of the growing rectangle as a sum: Namely they treated the growing rectangle as a composite number rectangle.

8- Varying Addends: n can be anything (Stacy). There are many different ways of writing the sum. With the functional notation, f(i)=anything, for all i. And f(i) is not necessarily equal to f(j) for any i≠j, where i, j denote the ordering number for the addends (areal units).

9- Unit Addends (Type II): The area of the polynomial rectangle is written as the sum of irreducible areal units. 4 PST came up with this usage. For instance, for the 2x+y by x+2y+1 rectangle, the unit addends are [x^2, x^2, yx, xy, xy, y^2, xy, xy, y^2, x, x, y].

10- "Boxes of the Same Color" type Addends: The area of the polynomial rectangle is written as the sum of the boxes of the same color. As an example, only 1 PST used the areal units [$2x^2$, 4xy, 2x, yx, $2y^2$, y] to generate the 2x+y by x+2y+1 polynomial rectangle.

There is one more type RUC, in between additive and multiplicative which I named Pseudo Multiplicative type RUC. This occurred for the "Area of the Boxes of the Same Color as a Product" in dealing with polynomial rectangles made of color tiles. For instance, the x+1 by 2y+3 rectangle has 4 boxes (x by 2y, x by 3, 1 by 2y, 1 by 3) of the same color. Nicole, Stacy, and John's products were x·2y, x·3, 1·2y, 1·3; i.e., of multiplicative nature. With the relational notation, these linear units, namely the length and the width of each "box" can be written as (x,2y), (x,3), (1,2y), (1,3). However, Ben and Ron's areas as a "product" for the same boxes were 2·xy, 3·x, 2·y, 3·1. In other words, the first term of each "pseudo-product" is a coefficient serving as a counting number indicating how many there are of each irreducible areal unit: Though written as a "product", Ben and Ron's expressions are of additive nature.

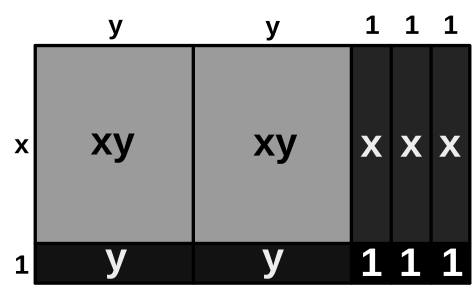

Figure 2. Polynomial Rectangle of Sides x+1 by 2y+3

Relational Notation & Mapping Structures: A Data Analysis Framework

I have decided to extend Behr et al.'s notation in such a way as to cover identities that equate summation and product expressions of representational quantities. Students may not realize what we, as adults, take for granted in a multiplicative situation: that commutativity is one of the properties of multiplication. Yes, numbers commute when the binary operation under consideration is multiplication. However, though produce the same result, a×b and b×a may have different algebraic and geometric interpretations for students. In other words, although the commutative property states that order does not matter in multiplication, it may matter for some students. The relational notation I am proposing thus, naturally follows these ideas, as an ordered pair (a,b) is in general different from (b,a). (They are the same only if a=b). Even though I use the notation (a,b) to denote an ordered pair of linear units, for these research participants, (a,b) did not have a different meaning from (b,a) therefore, in the remaining part of this manuscript, the ordered pairs (a,b) and (b,a) are to be considered equivalent.

It remains to describe a relational notation for addition, as well. In this case, once again I am not going to worry about commutative property of addition for the same reason as stated for multiplication. However, I am going to reserve square brackets [ ] for addition, which will be different from the multiplicative situation. Moreover, for the multiplicative situation, we have ordered pairs while for the additive situation, we have ordered n-tuples where the integer n in general being greater than 2. For instance, the ordered n-tuple $[a_1, a_2, ..., a_n]$ denotes the areal units a_i, $1 \leq i \leq n$, generating the area of the growing rectangle under consideration.

There must be an agreement of the ordered pair (a,b) of linear units and the ordered n-tuple $[a_1, a_2, ..., a_n]$ of areal units. How can we reconcile these two? At this moment, it's mapping structures which comes to rescue. Focusing on the x+1 by 2y+3 polynomial rectangle example I gave above, the multiplication operation, which behaves as a function, as a mapping, can be represented using a functional notation such as f: (x+1,2y+3) → 2xy+3x+2y+3. Here, f denotes the multiplication operation which maps the linear units, x+1 and 2y+3, which can be thought of as a combination of irreducible linear units, into the corresponding areal unit, namely 2xy+3x+2y+3, which is also the same as the area of the polynomial rectangle itself. In other words, f acts on the ordered pair (x+1, 2y+3) of linear units and maps it into the areal unit 2xy+3x+2y+3. This operation can also be written as f(x+1,2y+3)=2xy+3x+2y+3.

Similarly, the addition operation behaves like a function, like a mapping, acting on irreducible areal units or a combination of those. For instance, the function g, which represents the addition operation, acts on the ordered 10-tuple [xy, xy, x, x, x, y, y, 1, 1, 1] of areal units and maps it into the areal unit 2xy+3x+2y+3. Using a functional notation, this can be written as g: [xy, xy, x, x, x, y, y, 1, 1, 1] → 2xy+3x+2y+3, or, with the equality g[xy, xy, x, x, x, y, y, 1, 1, 1]=2xy+3x+2y+3. In other words, though they act on different types of representational quantities, the mappings f and g agree on one thing: That one thing is nothing but the fact that their images coincide. This is the essence of what is meant by "identity" in this research project. Area as a product coincides with the area as a sum eventually, thanks to these mapping structures.

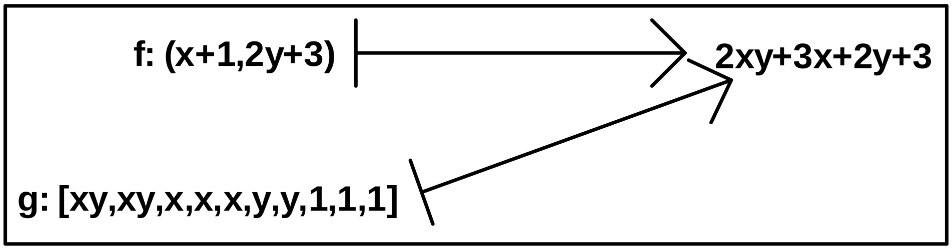

Figure 3. Relational Notation & Mapping Structures

Students and teachers should pay attention to the identification and the coordination of representational units of different types associated with color cubes and tiles, which are critical aspects of quantitative reasoning. Moreover, mapping structures serve as a bridge in between area as a sum and area as a product concepts in helping students and teachers make sense of identities on integers and polynomials: Why does the LHS have to be equal to the RHS?

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