Tackling Epistemological Problems (TEP)

A didactical engineering to break through
the amazing world of regular polyhedra

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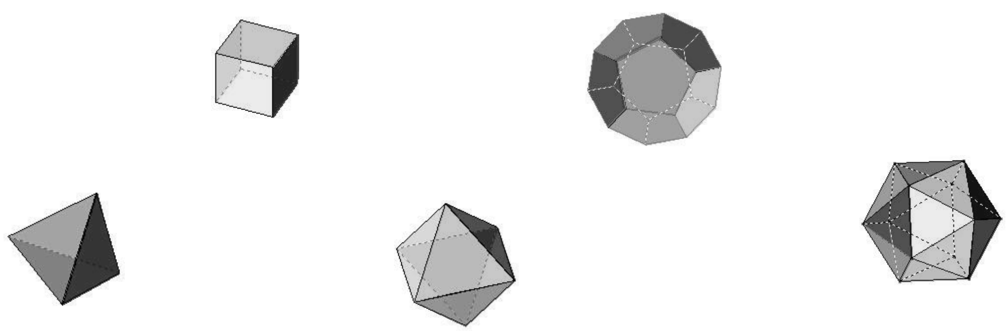
Abstract
An epistemological problem is a problem which is significant in the development of maths itself and it´s significant for each person to build his own mathematical knowledge.
There are infinite regular polygons but **only 5** regular polyhedra . Just this fact makes the study of the five regular polyhedra an epistemological problem which, no doubt, deserves to be studied.



"**Tackling**" puts the emphasis on the fact of facing (with different tools) this type of problems ; the obvious goal is to solve them but we shouldn´t discard unsolved problems which might be rich in maths.
In this new approach called **TEP** there are 3 key words : multirepresentation, approximation and integration. Multirepresentation is what allows to build concepts in maths, approximation is the road that leads to exactness and integration is what makes maths powerful.
The work will start at a very concrete level manipulating and playing with plastic tube 3D models provided by the teacher, leading the participants to the most abstract level of exact calculation, using quadratic irrational numbers.

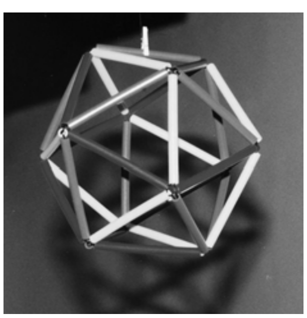
INTRODUCTION

Space geometry is indeed an epistemological obstacle.
It is amazing that in textbooks about geometry there are lots of problems about cubes, prisms and pyramids but very seldom about the regular dodecahedron and the icosahedron in spite of their indisputable epistemological value.

Why?

Science, even mathematics, starts with experimentation and in geometry the starting point of experimentation is representation and in particular in space geometry this starting point should be 3D models. To represent a 3D object with a plane figure is a very exigent task that can easily exceed the difficulties of reasoning on a course of space geometry.
In this sense, try to teach space geometry, just with static and plane drawings, becomes a didactical obstacle for the learning of this geometry.
We need to use 3D models and also dynamic figures (as those furnished by the software of dynamic geometry CABRI 3D) to reach the essential visualization to be able to find properties, to build concepts and make reasoning.
If we lack of a proper representation of the geometrical object we are blocked at the very beginning of all the process ; even though, plane and static figures are poor representations of 3D objects , they are always necessary, but we must realize that they are cognitively very demanding and if we try to use them too soon, instead of favoring the learning of space geometry, they make it harder.
The icosahedron and the dodecahedron (that are rich in maths) are practically unknown among our high school students . It is hard to believe how they remain unstudied and almost ignored. This workshop tries to break through their fantastic world.

THE DEVELOPMENT OF THE WORKSHOP



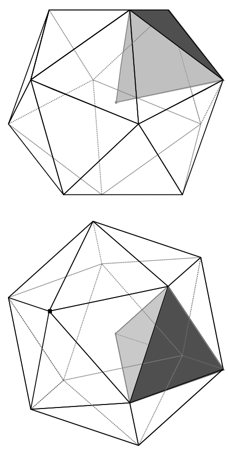
Employing plastic tube 3D models of the regular polyhedra the participants would count the number of faces, edges and vertex of the regular polyhedra and find the relation between them.
This relation could be extended to other type of non regular polyhedra and the search of a non-eulerian body could be a problem to be proposed.

We are going to focus on working their volumes (a property that doesn´t exist in plane geometry)

THE VOLUME OF THE REGULAR ICOSAHEDRON

THE APPROXIMATE METHOD

A regular icosahedron has 20 faces,12 vertex and 30 edges
(counted on the plastic 3D models of the former picture)



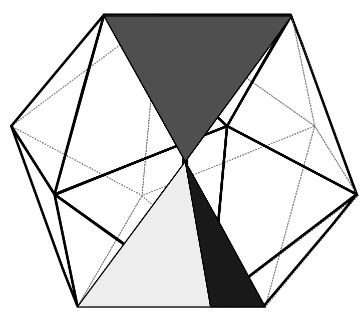
The center of the icosahedron is equidistant from all the vertex.
Thus, the icosahedron can be broken down into pyramids having as base a face of the icosa and as vertex the center of the icosa.
You can see one of those pyramids in this drawing.
There are as many pyramids as faces of the icosa , that is 20 .

If we manage to calculate the volume of one pyramid, multiplying by 20 we would have the volume of the icosa

The area of the base of the pyramid (an equilateral triangle) is no problem, but the height of it, it´s harder . The first method we could think of, is to measure with a ruler the height of this pyramid.

But how can we measure the height of a pyramid?

Since two opposite faces of an icosa are parallel , the distance between them is two times the height of a pyramid as we can see in this figure.
In a 3D cardboard model pupils can easily measure this distance and dividing by 2, they find the height of the pyramid.



Then, they can calculate the volume of one pyramid by the formula:

$$V_{1\,Pyramid} = \frac{A_{Base} \times h_{Pyramid}}{3}$$

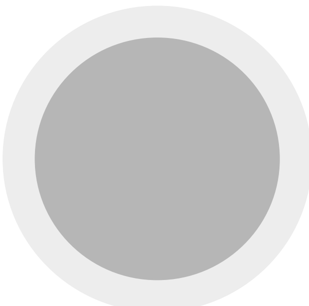
and multiplying by 20 (the number of pyramids)

we get the approximate value of the volume of that particular cardboard icosahedron

THE EXACT METHOD

Since the icosahedron is regular, knowing just the size of the edge everything else is determined.
Thus, we would calculate the volume of the icosa having the size of its edge as sole data.

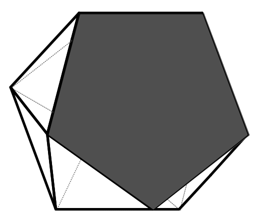
THE THREE SPHERES



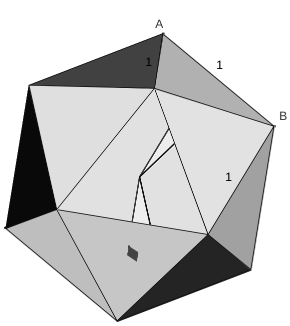
The 12 vertex are equidistant from the center of the icosa. Then, there is a sphere (the yellow one) passing through all the vertex.
The 20 faces are also equidistant from the center of the icosa . So there is a second sphere (the green one) also having as center the center of the icosa and tangent to all the faces.
There is a third sphere (not drawn here) with the same center that passes through the midpoints of all the edges.
The first sphere is the circumscribed sphere, the second one the inscribed sphere and the third one we would call it the middle sphere.

THE PENTAGONAL SECTION

Each vertex belongs to 5 edges.
For each vertex there is a regular pentagon as a section .
Since the icosa has 12 vertex there are 12 pentagons as this one, in each icosa.

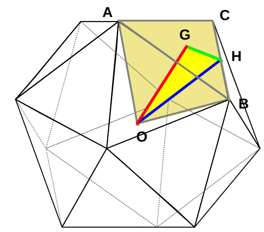


THE KEY TRIANGLE



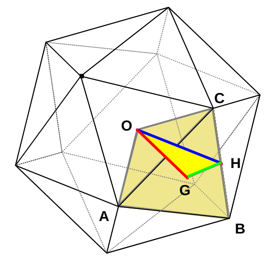
We have the right angled triangle OGH .
OH is the radius of the middle sphere and is equal to half the diagonal of the pentagon.
G is the center of the face so GH is one third of the height of an equilateral triangle. Thus if we know two sides of a right angled triangle we can apply Pythagoras Theorem to calculate OG , which is the height of one of the pyramids that form the icosa.

We have seen already that the icosa can be broken down in 2O equal pyramids as this one.
We can see again the key triangle OGH.
Lets take that the length of the icosa's edge is **1**
Being H the midpoint of BC then
AH is the height of an equilateral triangle



$$\Rightarrow \ \mathrm{AH} = \frac{\sqrt{3}}{2}$$

Being G the center of ABC



$$\mathrm{GH} = \frac{1}{3}\mathrm{AH} \ \Rightarrow \ \mathrm{GH} = \frac{\sqrt{3}}{6}$$

We have seen already that OH is that the hypotenuse OH is half the pentagon's diagonal and we know that the length of a regular pentagon´s diagonal is :

$$\frac{1+\sqrt{5}}{2} \quad \Rightarrow \quad \mathrm{OH} = \frac{1+\sqrt{5}}{4}$$

We have a right angled triangle and we know OH and GH .
We can apply the Pythagoras Theorem to calculate OG , height of the pyramid.

$$\mathrm{OG}^2 = \mathrm{OH}^2 \quad \mathrm{GH}^2 \qquad\qquad \mathrm{OG}^2 = \left(\frac{1+\sqrt{5}}{4}\right)^2 - \left(\frac{\sqrt{3}}{6}\right)^2$$

we operate $\mathrm{OG}^2 = \dfrac{6+2\sqrt{5}}{16} - \dfrac{3}{36} \ \overset{\text{simplifying}}{\Longrightarrow} \ \mathrm{OG}^2 = \dfrac{3+\sqrt{5}}{8} - \dfrac{1}{12}$

which makes $\mathrm{OG}^2 = \dfrac{7+3\sqrt{5}}{24} \ \Rightarrow \ \mathrm{OG} = \sqrt{\dfrac{7+3\sqrt{5}}{24}} \qquad \mathrm{OG}^2 = \dfrac{14+6\sqrt{5}}{48}$

$$\mathrm{OG}^2 = \frac{(3+\sqrt{5})^2}{16.3} \ \overset{\text{taking square root}}{\Longrightarrow} \ \mathrm{OG} = \frac{3+\sqrt{5}}{4\sqrt{3}}$$

If **AB = 1** the height of the base (an equilateral triangle) is $\dfrac{\sqrt{3}}{2}$

Therefore the Area of the base is: $$A_{\text{Base}} = \frac{1\times\frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{4}$$

Now we will calculate the volume of one pyramid $$V_{1\,\text{Pyramid}} = \frac{A_{\text{Base}}\times h_{\text{Pyramid}}}{3}$$

$$V_{1\text{Pyramid}} = \frac{\left(\frac{\sqrt{3}}{4}\right)\left(\frac{3+\sqrt{5}}{4\sqrt{3}}\right)}{3} = \frac{\left(\frac{\cancel{\sqrt{3}}}{4}\right)\left(\frac{3+\sqrt{5}}{4\cancel{\sqrt{3}}}\right)}{3}$$

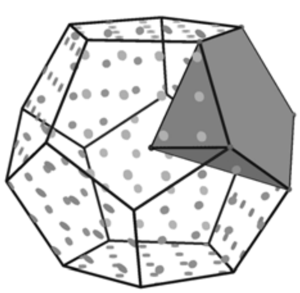
$$\Rightarrow \qquad V_{1\text{Pyramid}} = \frac{3+\sqrt{5}}{48}$$

But since we have 20 equal pyramides we have to multiply by 20

$$V_{\text{ICOSA}} = \frac{3+\sqrt{5}}{48}\times 20 = \frac{3+\sqrt{5}}{12}\times 5 \qquad V_{\text{ICOSA}} = \frac{15+5\sqrt{5}}{12}$$

If we generalize then the volume of an *"a"* edge icosahedron is

$$\boxed{V_{\text{ICOSAHEDRON}} = \frac{15+5\sqrt{5}}{12}\,a^3}$$



If we work in a similar way with the pyramid in the picture at left, we could find the volume of the Dode.

We should get $$V_{DODE} = \frac{15+7\sqrt{5}}{4}a^3$$

Both the volume of the Icosa and the Dode are quite similar, pretty simple quadratic irrationals numbers, as we can see !
But the icosahedron and the dodecahedron have a lot of more of surprises reserved for us!