What are students thinking as they solve open-ended mathematics problems?

Mary Margaret Capraro
Assistant Professor of Mathematics Education, Texas A & M University
Robert M. Capraro
Associate Professor of Mathematics Education, Texas A & M University
Victor V. Cifarelli
Associate Professor of Mathematics, University of North Carolina, Charlotte

Abstract
This paper reports the results of a study of college students' problem solving. Participants (N=73) came from undergraduate math education classes at a southern university in the US. Earlier studies (Capraro, Cifarelli, Capraro, & Zientek, 2006; Cifarelli & Cai, 2005) found that the use of open-ended mathematics problems with secondary mathematics education students enabled them to develop and stretch their conceptual understanding. The current study examined the following questions: 1. Can the same tasks be used with middle grades education majors to achieve similar benefits? and 2. Can the benefits be maintained by providing instruction via an on-line format? Students solved a series of open-ended tasks and submitted their solutions as part of an on-line class using WebCT. Sixteen of the students participated in small group discussions of their solutions. Data sources included the on-line self reports and the researchers' field notes from observations of those who engaged in the discussion groups. While there was some compatibility between these results and the prior studies, the students did not demonstrate high level of mathematical sophistication in their solutions when compared to students of the earlier studies. However, the students in the discussion groups demonstrated more sophisticated mathematical activity than those using only the on-line format. The results suggest challenges that must be met in order to effectively implement open-ended problem solving in classrooms.

Introduction

Solving problems is a complex task that is essential to the teaching and learning of mathematics (NCTM, 2000). In our interviews with pre-service mathematics teachers in problem solving classes, one student reported the ways she proceeds when she encounters new mathematics problems: "I go straight to the quickest way to get an answer. The reason I probably do this is because I had teachers who would not wait for me to solve a problem in a different way". This comment indicates how a teacher's use of narrowly defined problems may influence how students come to view what is valued as a mathematical activity. To help students develop into good problem solvers, mathematics educators should provide both well-structured problems and open-ended problems. The characteristics of well-structured problems are that the problem conditions are explicitly stated within the problem statement. This is compared to open-ended problems (Becker & Shimada, 1997), where the goals are not explicitly stated within the problem statement necessitating the learner to develop specific goals for her/his actions. While it is necessary to provide students with rich problem solving experiences that include both well-structured and open-ended problems, this study focuses on strategies for implementation of open-ended problem solving for middle-grades, pre-service teachers.

Benefits of Open-ended Problem Solving

Open-ended problem solving provides a free and supportive learning environment for students to develop and express their mathematical understandings. The educational benefits for students are many. Since open-ended problems allow for different correct solutions, each student has opportunities to obtain her/his own unique solutions. Every student can respond to the problem in some significant way. It is important for every student to be involved in classroom activities and lessons should be understandable for every student. Students have more opportunities to make comprehensive use of their mathematical knowledge and skills. With many different solutions, students can choose their favorite strategies to obtain answers and create their unique solutions. Teachers in turn are able to conduct rich discussions with students that involve the various strategies students used to solve problems. Through comparing and discussing, students are motivated to give other students reasons for their solutions. This affords great opportunities for students to develop their mathematical thinking. Rich experiences allow students to have the pleasure of discovery and receive approval from fellow students (Sawada, 1997).

Research on Open-Ended Problem Solving

There are at least two ways that mathematics education researchers have viewed open-ended problem solving as an exploratory process. Cognitive psychologists see problem openness in objective terms, choosing to focus on the explicit structure of tasks that can be manipulated and then see how students cope (Mayer, 1985, Reed, 2000). According to this approach, tasks can be structured with embedded mathematical properties so the researcher has a reasonable expectation of the possible discoveries that students can make; hence, the focus is on how the students will work inductively to discover mathematical properties. Foundationally, the transfer of learning studies of the 70s and 80s exemplified this view of problem structure. The implications are that students will engage in inductive exploration, using trial-and-error strategies to develop their patterns. In contrast, our approach focuses more on the solver's evolving sense-making as he/she engages in problem solving situations. Hence, we observe closely the solver's initial interpretations and how she/he proceeds to develop goals for action. The challenge for us then involves trying to see the solver's point of view and focus on the questions that become important as she/he proceeds.

Goals and Purposes

Prior studies (Capraro, Cifarelli, Capraro, & Zientek, 2006; Cifarelli & Cai, 2005) found that having secondary mathematics education students solve open-ended mathematics tasks provided them with unique learning opportunities to stretch and extend their conceptual boundaries of understanding. From the results of these studies, we became interested in determining if and how middle grades pre-service teachers could benefit from solving open-ended problems. The current study looked to build on these results by examining the following research questions: 1. Can the same kinds of open-ended tasks be used with middle grades pre-service teachers as learning opportunities, to achieve similar benefits? and 2. Can the benefits be maintained by providing instruction via an on-line format. Hence, the current study examined the problem solving actions of students in two instructional settings.

Subjects and Tasks

The participants were middle school pre-service teachers (N = 73) who were enrolled in *Problem Solving for Middle School Teachers*. This course is based on the four main steps of Polya's problem solving strategy. The 73 participants were divided into two groups. Group 1 students (n=57) solved the task while working alone submitting their solutions through WebCT. Group 2 students (n=16) also submitted their solutions online but were engaged in cooperative group activities .where they shared solutions with peers through small group discussions. All students were again presented with the task at the end of the semester.

Students in the study solved a set of open-ended tasks. In this paper, we focus on one of these, a number array task (Figure 1). Students were asked to find as many relationships as possible among the numbers and to record the various relationships and tell what they were thinking as they explored the array.

Figure 1: The Number Array Task

The following table of numbers was produced in accordance with a certain rule and has many rich relationships. Study the arrangement of numbers in the table and find as many relationships as possible among the numbers.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

Results

Table 1: Mathematical Relationships	**Gr. 1** n=57 $\bar{X}$=4	**Gr. 2** n=16 $\bar{X}$ = 9	**Prior Studies**
Relationships about the spatial arrangement of numbers			
1. Numbers in columns and row are multiples of 1,2,3,4, …, 10	**25**	**14**	**X**
2. Numbers in each row/column are arithmetic progressions	**35**	**16**	**X**
3. Numbers on the main diagonal are square numbers	**28**	**16**	**X**
4. Numbers are symmetric about the main diagonal	**18**	**16**	**X**
5. Corresponding rows and columns are identical.	**13**	**16**	**X**
6. Diagonal from to left to bottom right contains consistent a pattern of increasing by an incremental even or odd amount.	**9**	**15**	**X**
7. Each diagonal begins with an even or amount.	**5**	**14**	**X**
Relationships about the sums of the numbers			
8. The sum numbers in each row/column is a multiple of 55.		**1**	**X**
9. Sum of individual rows of the table is the row of the sum		**2**	**X**
10. Difference in sums of numbers on opposite corners of 2x2 blocks is 1.			**X**
11. The sum of the two numbers in a row or column located symmetrically about a pivot is two times the pivot number.		**5**	**X**
12. The difference between the sums of the numbers on opposite corners of a rectangular block having row and column length m and n is m n.		**10**	**X**
12. The sum of numbers in square blocks along the main diagonal can be computed using row and column entries.		**13**	**X**
13. Sum of entries in block = (sum of rows)•(sum of columns)			**X**
Relationships about the products of the numbers			
14. The number in the m^{th} row and n^{th} column is m n		**14**	**X**
15. The products of the diagonal end numbers in a block are equal.		**4**	**X**
16. For a square, the product of the diagonals is equal.		**1**	**X**
Relationships about number sequences			
17. The individual sums of each 2x2 block is a square		**7**	**X**
18. The sum of entries in all 2x2 blocks along the top row is 9+6(n-2), n=right column number of the block			**X**
20. The sum of entries in all 3x3 blocks along the top row is 36+18(n-2), n=right column number of the block			**X**
21. The sum of entries in all 4x4 blocks along the top row is 100+40(n-2), n=right column number of the block			**X**
22. The sum of entries in any NxN block along the main diagonal equals the square of the sum of column numbers			**X**
Total	**133**	**148**	

In solving the Number Array task, the students found a variety of relationships, many of which were similar to those in the earlier study. Students in Group 1 identified an average of 4 relationships with individual responses ranging from 1 to 7 relationships. In comparison, students in Group 2 identified an average of 9 relationships, with individual responses ranging from 1 to10 relationships. However, the students' list of relationships did not appear to be as mathematically sophisticated as those developed by the students of the prior studies. The results are summarized and compared to the earlier studies in Table 1.

Discussion

In discussing the results of the study, we return to the original research questions.

1. Can the same tasks be used with middle grades pre-service teachers to achieve similar benefits?

Open-ended tasks need to be carefully constructed so the mathematical level is appropriate to the students' levels of understanding. In this case, the tasks worked pretty well for the Group 2 students despite their lower level of formal preparation as compared to the students from earlier studies. The group discussions gave students the needed peer interaction to develop a range of mathematical relationships.

2. Can the benefits be maintained by providing instruction via an on-line format?

While an online format appears to hold promise as a means for teaching mathematical problem solving, the results suggest that the sole use of WebCt as an instructional format does not adequately address the variety of advanced mathematical thinking processes that students need to solve open ended problems. The comparatively weaker performance of the Group 1 students can be attributed to at least two reasons. First, the students' were clearly uncomfortable as they solved the open-ended problem and this impacted their solutions. As we noted in the Introduction, students are accustomed to working problems where a specific answer can be found; the student's focus then becomes finding the quickest way to produce an answer, an approach that she/he believes is expected and valued by the classroom teacher. A second reason why the Group 1 students lagged behind their Group 2 counterparts concerns their lack of interaction with other students, as they proceeded from initial sense-making to developing goals and strategies for completing the task. The Group 2 students had on-going opportunities to try out their individual solution ideas through discussion with peers, the result of which they were able to stretch and extend their classifications of mathematical relationships. Hence, while on-line instruction has shown promise as an instructional tool, the results indicate that successful implementation may require teachers to consider carefully the novel requirements of open-ended problems in introducing these problems to students. In addition, effective implementation may involve retaining some aspects of current classroom practices such as teacher mediated small group discussions.

Summary and Conclusions

While there is some compatibility between these results and those of the prior studies, students of the current study did not identify many of the sophisticated mathematical relationships as students from the earlier studies. It is not surprising that the middle grades pre-service teachers overall performance lagged behind the students in the earlier studies who came to the tasks with more formal mathematics preparation. That being said, the results indicate the merits of using small group discussion activities in an on-line class. In particular, the students who received on-line instruction only (Group 1) identified relationships based only on the spatial arrangement of the numbers. In contrast, the students who had opportunity to share their constructed relationships with peers (Group 2) fared much better, identifying relationships that involved mathematical operations on numbers in the array. The results indicate that an on-line only instructional format does not lend itself to generating a variety of creative responses in its present format. However, the Group 2 results suggest that an on-line approach combined with supplemental discussion activities provide a useful learning environment for students to develop their explorations and discoveries.

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