Findings from Two Countries: Regarding Prospective Teachers' Knowledge of Addition and Division of Fractions

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Abstract

This study reports the initial findings by two collaborating mathematics educators, one from the United States and one from Northern Ireland, who are evaluating their prospective elementary teachers on the depth of their understanding of rational numbers. The literature suggests that in addition to the ability to explain the procedures of operations with fractions, elementary teachers need a deep conceptual understanding of rational numbers before they can effectively transfer their knowledge to their students. While traditional measures can easily assess whether these future teachers have the ability to perform operations and solve real world problems with fractions, a way to determine whether these future teachers have a deep understanding of addition and division of fractions is to determine if they can actually create appropriate classroom problems requiring these operations. Thus to measure the level of understanding, the two mathematics educators asked their prospective elementary teachers to create real life problems illustrating the addition and division of fractions. This paper reports the initial qualitative analysis of these student responses and discusses similarities as well as interesting differences in the students' responses.

Introduction

The level of the mathematics understanding of rational numbers achieved by prospective elementary teachers has been a concern in the research of mathematics education for at least two decades [1], [2], [3], [4]. Ball's investigation [1] of the mathematical understandings of prospective teachers found their knowledge often to be thin and rule-bound. Other researchers also concluded that prospective teachers need to revisit the mathematics that they have previously learned to strengthen their understanding so that they will be able to teach the mathematics to their future students. The Conference Board of Mathematical Sciences (CBMS) specifically notes that strengthening rational number knowledge and rational number sense are "absolute essential components" in the preparation of middle grades mathematics teachers. Besides being able to explain procedures, future teachers need to develop a sufficient depth of understanding in order to teach in the middle grades. The CBMS further suggests that prospective teachers should be able to write problems that can be solved by a particular arithmetic operation since prospective teachers who have this deep understanding of fractions will be more likely to help their future students develop their own understanding.

Parker's research [5] specifically studies the knowledge of prospective elementary teachers in the area of addition of fractions while emphasizing the critical role of the unit of reference. This research extends Parker's problem posing design to include both addition and division of fractions.

Methodology

During the spring semester of 2007, 34 United States (US) and 6 Northern Ireland (NI) prospective elementary teachers were given the task of preparing problems involving addition and division of fractions that would be appropriate for future elementary students. The difference in sample size in the two countries is due largely to the lack of availability of Northern Ireland students when the data was gathered.

The Northern Ireland students are all in the final year of a four year Bachelor of Education degree that prepares them for teaching in elementary (primary) schools. Each of the students specializes in mathematics and has already spent approximately one quarter of their degree time studying mathematics. They have also spent the equivalent of one quarter of their degree time teaching in elementary schools and have classroom experience in teaching mathematics.

The US students are not as advanced in working toward their four year degree in elementary education. The US students are at various stages in their career and consist of 2 freshman, 20 sophomores, 7 juniors, 1 senior and 4 post graduates (who were returning to study elementary education). When the data were gathered for the study, the US students were in the final weeks of completing their first required mathematical content knowledge course that included content for Kindergarten through grade 8.

A limitation of the study, therefore, is the difference in the background between the US students and those in Northern Ireland. Another limitation of the study is that the US students' data were collected from their unit exams on fractions, decimals, and percents, and thus their ability to create problems was evaluated immediately following instruction. The NI data was collected from a separate test administered in their senior year. It should also be noted that the US students had responded to the addition problem in this study prior to instruction and received feedback on their responses. While the US students did not previously write a problem for division of fractions, they had studied models of fraction division with number lines and pattern blocks with different referent wholes.

Procedure

The future teachers were required to demonstrate their knowledge of addition and division of fractions by creating problems that would be appropriate for elementary students. The first problem required them to add two fractions whose sum is greater than one and the requirement for the second problem was to divide a mixed number by a fraction. The following problems were completed by the 40 prospective teachers.

1. Write a story problem where students in the elementary grades would add ¾ + ½ to complete the problem. [5]

2. Write a story problem that shows the meaning of $2\frac{1}{2} \div \frac{1}{2}$.

While the US students were given these problems as part of an exam on their knowledge of fractions, decimals, and percent, the Northern Ireland students were not graded on their responses to the questions. The Northern Ireland students, however, were instead given 20 minutes to write questions for four problems regarding fractions, decimals, and percentages.

After the students completed the problems, their respective professor/researcher categorized their problems as either acceptable or unacceptable. Both principal researchers asked one other researcher from their country to independently categorize the questions. There was a 100% inter-rater agreement with the two Northern Irish researchers, and 96% inter-rater agreement with two US researchers. The US researchers then shared their non-matching responses with the Northern research team to obtain a majority decision. The small number of non-matches (4 for US) was then discussed among the researchers on the team. As a result of the discussion there was a final (100%) agreement on the acceptable and unacceptable classifications of all the problems. The Northern Ireland team requested that the US team review three responses in the addition problem, which brought forth a discussion on the emphasis on the referent whole.

Within the acceptable items, similar categories of acceptable and unacceptable responses emerged from both countries. The results of the classifications follow.

Results *Addition of Fractions*

For this study, 70% percent (28 out of 40) of the US students and none of the prospective teachers from Northern Ireland (NI) provided acceptable responses in designing a real-life fraction addition problem for the sum of one half and three-fourths. When the US students completed a pretest with the same problem prior to instruction, only 20% wrote acceptable responses with many of the inappropriate problems not referring to equivalent wholes. During instruction, the concept of equivalent wholes was emphasized, and as a result the US students showed an increase in their ability to create an appropriate problem for this additive scenario. The Northern Ireland students had not explicitly discussed referent wholes in class prior to completing this question.

Of the 28 US acceptable problems, 20 were related to food and18 specifically involved recipes. The problems often used both of the fractions as measurement units in a given recipe and then posed the question of finding the total amount of ingredients in the recipe. An example of a typical recipe problem follows:

Today we are going to make cookies. We need to add ½ cup of sugar and ¾ cup of brown sugar. Together, how many cups do the two sugars add up to?

Another appropriate type of problem that students wrote was also a measurement model of their experiences with running.

While I was running, I kept track of my miles. I ran ½ mile on Monday morning and ¾ mile on Monday night. How far did I run on Monday?

With these types of categories, it was not required for students to consider that the sum of ½ and ¾ is greater than one. In these measurement situations, they did not need a referent whole, as did the students who considered the sum being greater than one, and thus referred to two wholes. Only four students created appropriate problems that made an appropriate reference to two equivalent wholes. The following example shows a response in this category.

You have 2 pizzas that are exactly the same size. You give one pizza to your friend and you decide to see who can eat the most. You eat one half of your pizza. Your friend eats ¾ of his pizza. How much pizza did you both eat all together?

This student was very careful to designate that both pizzas are equal in size since the equivalent wholes were addressed during instruction. In other similar problems where students used two pizzas, one could argue that possibly one person has a different size pizza than the other person hence finding the sum of ¾ + ½ would not be an appropriate problem.

The same issue arose with three of the Northern Ireland students' solutions as they did not make it explicit that the "wholes" were equivalent, for example:

Sarah eats half of a chocolate cake and three quarters of an orange cake. How much cake does she eat altogether?

In all three of these cases an underlying assumption was made that the two chosen objects, in this case the chocolate cake and the orange cake are equivalent in shape and size and that the answer of one and one quarter cakes would have some significance. The students did not articulate this assumption in their writing and were not challenged on its existence.

There were also US unacceptable responses similar to the NI responses above in that they did not make reference to referent wholes.

If the girls eat ½ of a blueberry pie and the boys eat ¾ of an apple, how much pie is left over?

The problem, as posed, required a response of 1 ¼, but the answer to this problem is three-quarters. Another type of an unacceptable response from Northern Ireland was:

There are two bars of chocolate with 16 squares in each. Jane eats half of one bar and Peter eats three quarters. How many squares are left to share with Angela?

Here the pupil attempting this question is not actually required to carry out the calculation ¾ + ½. Instead they can find one half of 16, then three quarters of 16, add the results and take this from 32 to obtain the answer. It is also not clear in this question if Peter is eating three quarters of the same bar as Jane or a of the second bar although the underlying assumption is that it is of the second bar.

In the next phase of the project, the researchers will interview students to attempt to determine if the students are actually aware that they are making the assumption concerning equivalent wholes. The researchers will also attempt to determine if students wrote measurement problems to avoid writing a problem that needed to refer to equivalent wholes.

Division of Fractions

2. Write a story problem that shows the meaning of $2\frac{1}{2} \div \frac{1}{2}$.

For this question 83% of the Northern Ireland prospective teachers provided acceptable answers. A typical response of half of the students is below.

If there are two and a half pizzas and each child received half a pizza, how many children share it?

An additional third of the NI students provided answers that showed an acceptable insight to the question, but also went a little further in terms of what they asked the pupils to do, for example: *Simon has two and a half pizzas. He wants to share his pizzas with his friends and he wants each person to get half a pizza. How many friends can he share with?*

Clearly here the pupil has to carry out the calculation, but then needs to subtract one from the answer to find the number of Simon's friends, i.e. excluding Simon himself.

The only division problem response that was deemed unacceptable for the NI students is provided below,

The group bought two litre bottles of water and a 500ml bottle. When everyone had had a cup there was still half left over. How many mls were left?

It is not clear here when the student states that there was still half left over if they are referring to half a litre or half the total amount of water. If it is the former, there is a possibility that they may have read the question as a subtraction problem. If it was the latter, then this shows a total lack of understanding of this division problem. A cultural difference is noted in this problem, as the US students did not use the metric measurement system for their problems because of the different measuring system.

The US results show 21 acceptable responses (62%), 11 unacceptable responses (32%), and 2 students who did not respond (6%) [possibly because they did not have time to complete the item since it was part of their unit exam]. US students were not pre-tested on the division problem, but were instructed using number line models and pattern block referent wholes for area models. The instruction also situated problems in contextual settings that were interesting to the students.

Similar to the addition stories, US students referred to food in 55% of the acceptable responses. Pizza was the most referenced food and some problems even included drawings of cookies, butter, bagels and other foods.

During US instruction, an example was presented to determine how many ¾ yard bows could be made from 5 yards of ribbon [6], and the instructor related the problem to the bows that are made for the mum corsages for their campus Homecoming celebrations. This particular example must have impressed students since six students created their division problem about creating bows. For example: *I am making bows. I have 2 ½ yards of material. Each bow uses a ½ yard of material. How many bows can I make?* Four students created similar measurement problems involving running. *Members of a track team ran 2 ½ miles all together. If each person on the team ran ½ of a mile, how many people ran the race?*

Several students in the class are members of the university track team, and thus related division of fractions to their own interests. The research suggests that making real world connections deepens understanding. [5] The frequently used measurement model also uncovered several student misconceptions as noted below. *You are making bows for a wedding. If the amount of ribbon is 2 ½ yards, how many bows can you make with ½ yard of ribbon?*

The student possibly was attempting to ask how many ½ yard ribbons could be made from 2 ½ yards, but did not clearly articulate an accurate division problem that would produce 5 full bows. This person possibly remembered the bow example discussed in class, but could not present the problem accurately.

Several misconceptions of subtracting a half or dividing in half rather than counting how many halves are included in 2 ½ were demonstrated. The following example shows 2 ½ being divided in half. *Laura was training for a track meet that she has on Saturday. She ran 2 ½ miles around her block, but ended up running half of that. How much did she run?*

In the next phase of the study, student interviews will be included to provide additional insight and clarify students' understanding in posing the problems.

Recommendations

The US students had a higher percentage of acceptable responses for the addition problem, while the NI students had a higher percentage of acceptable responses for the division problem. Since the US students had written addition problems prior to instruction, with only 21% writing acceptable solutions at that stage, the US students have shown a dramatic increase in their understanding following instruction.

Addressing how to accurately write addition problems prior to the division instruction also possibly made a difference in US students' ability to compose a relevant problem for division. In analyzing the problems that were written, students made connections from the contextual problems that were presented in class and frequently directly drew on that knowledge in creating their own division problems.

The Northern Ireland students had not explicitly addressed the issue of word problems for fractions previously and had only looked briefly at writing word problems for basic addition and subtraction of whole numbers. The fact that they performed so well on the division problem might be attributed to the fact that they were all senior math majors. The results should alert mathematics educators to the difficulties that prospective elementary teachers might have with these topics beyond the mere memorization of procedures, so that they can better prepare their students to deepen their knowledge in preparation for teaching.

The mathematics educators from both countries are sharing their teaching strategies and combining them to improve their lessons on addition and division of fractions. They also plan to share instructional strategies on other topics where their students exhibit difficulties. It is noted that the NI students are proficient at the computational methods and appear to struggle somewhat with conceptualizing these problems. This gives us an insight into the instructional opportunities we can share with each other in order to develop a deeper understanding by our students. The Northern Ireland team intends to incorporate similar problem posing situations into their instruction next year. The US educators are also interested in learning from the NI educators about their teaching strategies for division.

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