Power Series Explorations In Precalculus

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Abstract
This paper will start with an infinite geometric series and apply algebraic transformations to generate several new series, including transcendental functions. The purpose of this discussion is to prepare students for studying power series in calculus. This is often a challenge for second semester calculus students as they have to master the concepts of power series and apply the principles of calculus to power series. This process is often more successful if the students have a better understanding of power series.

Power Series Explorations In Precalculus

The expression $1 + x + x^2 + x^3 + x^4 + \ldots$ is an infinite geometric series with common ratio $r = x$ and first term $a1 = 1$. The infinite sum is $s = a1 / (1 – r) = 1 / (1 – x)$ if $|r| = |x| < 1$. That is, $1 + x + x^2 + x^3 + x^4 + \ldots = 1 / (1 – x)$, if $|x| < 1$.

If $p10(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^{10}$ is the tenth partial sum then we can confirm the equality by graphing $y1(x) = f(x) = 1 / (1 – x)$ and $y2(x) = p10(x)$ using the window $(-1, 1) \times (0, 10)$. Another method for obtaining this result is to divide 1 by $1 – x$ synthetically, using x as the zero of the divisor:

x |	1	0	0	0	0	0	….
		x	x^2	x^3	x^4	x^5	….
	1	x	x^2	x^3	x^4	x^5	….

$\Rightarrow 1 / (1 – x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

We can repeat this process for the rational function $g(x) = 1 / (1 + x)$.

We divide 1 by $1 + x$ synthetically, using $-x$ as the zero of the divisor:

-x |	1	0	0	0	0	0	….
		-x	x^2	-x^3	x^4	-x^5	…
	1	-x	x^2	-x^3	x^4	-x^5	…

$\Rightarrow 1 / (1 + x) = 1 – x + x^2 – x^3 + x^4 – x^5 + \ldots$

We can also consider $1 – x + x^2 – x^3 + x^4 – x^5 + \ldots$ as an infinite geometric series with common ration $r = -x$ and first term $a1 = 1$. The infinite sum is $s = a1 / (1 – r) = 1 / (1 – -x) = 1 / (1 + x)$ if $|r| = |-x| = |x| < 1$.

Another way of obtaining a series representation for $g(x) = 1 / (1 + x)$ is to use

$$g(x) = 1 / (1 + x) = 1 / (1 – -x) = f(-x) = 1 + -x + (-x)^2 + (-x)^3 + (-x)^4 + (-x)^5 + \ldots$$
$$= 1 – x + x^2 – x^3 + x^4 – x^5 + \ldots$$

If $q10(x) = 1 – x + x^2 – x^3 + x^4 – \ldots + x^{10}$ is the tenth partial sum then we can graphically confirm the equality by graphing

$y3(x) = g(x) = 1 / (1 + x)$ and $y4(x) = q10(x)$ over $(-1, 1) \times (0, 10)$.

Consider the function $h(x) = f(x) + g(x) = 1 / (1 – x) + 1 / (1 + x) = 2 / (1 – x^2)$. It seems reasonable that this can be represented by the sum of the corresponding

infinite series: $h(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + ... + 1 – x + x^2 – x^3 + x^4 – x^5 + ...$

$= 2 + 2x^2 + 2x^4 + 2x^6 + ...$ if $| x | < 1$.

We can confirm this by graphing y5(x) = h(x) and y6(x) = r12(x), where r12(x) is the partial sum of degree 12. Another way of obtaining a series representation for h(x) is to use:

$h(x) = 2f(x^2) = 2(1 + x^2 + (x^2)^2 + (x^2)^3 + (x^2)^4 + (x^2)^5 + ...) = 2 + 2x^2 + 2x^4 + 2x^6 + …$

Note that this is still an infinite geometric series with a1 = 2 and $r = x^2$.

Therefore the infinite sum is $h(x) = 2 / (1 – x^2)$.

Let us apply four basic transformations to f(x). if we use a vertical shift of one, then

$f1(x) = 1 + 1 / (1 – x) = 1 + 1 + x + x^2 + x^3 + x^4 + x^5 + ...$ or

$f1(x) = (2 – x) / (1 – x) = 2 + x + x^2 + x^3 + x^4 + x^5 + ...$

This is not a geometric series but it's clear that the infinite series representation is valid only if $| x | < 1$.

If we use a horizontal shift of one unit to the left, then

$f2(x) = f(x + 1) = 1 / (1 – (x + 1)) = -1 / x = 1 + (x + 1) + (x + 1)^2 + (x + 1)^3 + (x + 1)^4 + (x + 1)^5 + ..$

This is an infinite geometric series with a1 = 1 and r = x + 1. The infinite sum is $1 / (1 – (x + 1)) =$ $-1 / x$, where $| x + 1 | < 1$ ➔ $-1 < x + 1 < 1$ ➔ $-2 < x < 0$. We can confirm this graphically using y7(x) = f2(x) and y8(x) = s10(x), where s10 is the 10^{th} partial sum. If we use a vertical stretch of 2 then

$f3(x) = 2f(x) = 2 / (1 – x) = 2(1 + x + x^2 + x^3 + x^4 + x^5 + ..$

$= 2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + ...$ which is an infinite geometric series with a1 = 2 and r = x. Therefore the infinite sum is $2 / (1 – x) = f3(x)$. If we use a horizontal stretch of 2 then

$f4(x) = f(x/2) = 1 / (1 – x/2) = 2 / (2 – x) = 1 + x/2 + (x/2)^2 + (x/2)^3 + (x/2)^4 + (x/2)^5 + ...$

is an infinite geometric series with a1 = 1 and r = x/2 so the infinite sum is $1 / (1 – x/2) = f4(x)$, where $| x/2 | < 1$ or $-2 < x < 2$. We can confirm this graphically using y9(x) = f4(x) and y0(x) = t10(x) where t10(x) is the tenth partial sum.

Let us investigate the consequences of examining powers of f(x) and their associated infinite series.

$f(x) \cdot f(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + ...) \cdot (1 + x + x^2 + x^3 + x^4 + x^5 + ...)$ ➔

$1 / (1 – x)^2 = 1 + (1 + 1)x + (1 + 1 + 1)x^2 + (1 + 1 + 1 + 1)x^3 + (1 + 1 + 1 + 1 + 1)x^4 + …$

$1 / (1 – x)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + ...$ This is not a geometric series.

We can confirm this result graphically using $y1(x) = 1 / (1 – x)^2$ and y2(x) = p10(x), where p10(x) is the tenth partial sum. The series appears to converge for $| x | < 1$. We notice that the nth coefficient can be represented by n or nC1. Continuing this pattern, we have:

$[f(x)]^3 = f(x) \cdot [f(x)]^2$ or $1 / (1 – x)^3 = 1 / (1 – x) \cdot 1 / (1 – x)^2$ ➔

$1 / (1 – x)^3 = (1 + x + x^2 + x^3 + x^4 + x^5 + ...) \cdot (1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + ...)$

$= 1 + (1 + 2)x + (1 + 2 + 3)x^2 + (1 + 2 + 3 + 4)x^3 + (1 + 2 + 3 + 4 + 5)x^4 + …$

$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + ...$ This is not a geometric series.

We can confirm this result graphically using $y3(x) = 1 / (1 – x)^3$ and y4(x) = p10(x), where p10(x) is the tenth partial sum. the series appears to converge for $| x | < 1$. We notice that the nth coefficient can be represented by $n(n + 1) / 2$ or (n + 1)C2. Continuing this pattern, we have:

$[f(x)]^4 = f(x) \cdot [f(x)]^3$ or $1 / (1 – x)^4 = 1 / (1 – x) \cdot 1 / (1 – x)^3$ ➔

$1 / (1 – x)^4 = (1 + x + x^2 + x^3 + x^4 + x^5 + ...) \cdot (1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + ...) =$

$= 1 + (1 + 3)x + (1 + 3 + 6)x^2 + (1 + 3 + 6 + 10)x^3 + (1 + 3 + 6 + 10 + 15)x^4 + … =$

$= 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + ...$ This is not a geometric series.

We can confirm this result graphically using y5(x) = 1 / (1 – x)^4 and y6(x) = p10(x), where p10(x) is the tenth partial sum. The series appears to converge for | x | < 1. We notice that the coefficients are determined by a series. If we naively approach the problem of determining a formula for the coefficients, we can do an analysis of differences:

1 4 10 20 35 56 ➔ 3 6 10 15 21 ➔ 3 4 5 6 ➔ 1 1 1 ➔ the third differences are constant so the original sequence may be generated by a third degree polynomial. The nth coef is an^3 + bn^2 + cn + d. Using the first four coefficients, we obtain:

1 = a + b + c + d, 4 = 8a + 4b + 2c + d, 14 = 27a + 9b + 3c + d, 20 = 64a + 16b + 4c + d.

Solving, we have: a = 1/6, b = 1/2, c = 1/3, and d = 0. Therefore the nth coefficient is

(1/6)n^3 + (1/2)n^2 + (1/3)n = n(n^2 + 3n + 2)/6 = n(n+ 1)(n + 2) / 6 = n(n+ 1)(n + 2) / 3! = (n+2)c3.

$[f(x)]^5 = f(x) \cdot [f(x)]^4$ or $1 / (1 – x)^5 = 1 / (1 – x) \cdot 1 / (1 – x)^4$ ➔

$1 / (1 – x)^5 = (1 + x + x^2 + x^3 + x^4 + x^5 + ...) \cdot (1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + ...) =$

$= 1 + (1 + 4)x + (1 + 4 + 10)x^2 + (1 + 4 + 10 + 20)x^3 + (1 + 4 + 10 + 20 + 35)x^4 + …$

$= 1 + 5x + 15x^2 + 35x^3 + 70x^4 + 126x^5 + ...$ This is not a geometric series.

We can confirm this result graphically using y9(x) = 1 / (1 – x)^4 and y0(x) = p10(x), where p10(x) is the tenth partial sum. The series appears to converge for | x | < 1. We notice that the coefficients are determined by a series. If the coefficients follow a similar pattern to the previous series then the general coefficient should be n(n + 1)(n + 2)(n + 3) / 4! or (n + 3)c4. We can confirm this by letting n = 1, 2, 3, 4, 5, and testing the first five coefficients. We obtain 1, 5, 15, 35, and 70. Therefore if the coefficients are generated by a quartic polynomial then it must be the one we found since the solution is unique. The appearance of a factorial in the denominator leads us to several questions: suppose the coefficients are simply 1 / n!, or (-1)^n / n!, or some subset of these? We could ask similar questions if the denominator were simply n instead of n!. As we explore these questions we obtain some surprising results.

If the nth coefficient is 1 / n! then we have f(x) = 1 + x + x^2 / 2! + x^3 / 3! + x^4 / 4! + x^5 / 5! + …. Graphing this over [-2, 3] x [0, 20] we obtain a graph that appears to be an exponentially increasing function. If we compare it with y = e^x we obtain an almost perfect match.

If the nth coefficient is (-1)^n / n! then we have g(x) = 1 – x + x^2 / 2! – x^3 / 3! + x^4 / 4! – x^5 / 5! + ... Graphing this over [-3, 2] x [0, 20] we obtain a graph that appears to be an exponentially decreasing function. If we compare it with y = e^-x we obtain an almost perfect match. This should not be surprising as f(-x) = 1 – x + x^2 / 2! – x^3 / 3! + x^4 / 4! – x^5 / 5! + … = g(x).

If the nth term is x^(2n) / (2n)! then we have f(x) = 1 + x^2 / 2! + x^4 / 4! + x^6 / 6! + x^8 / 8! + …. Graphing this over [-3, 3] x [0, 10] we obtain a graph that appears to be symmetric with respect to the y-axis. We would expect this of an even function. It also appears to be a combination of exponentially increasing and decreasing functions. If we compare f(x) with y = cosh x = (e^x + e^-x) / 2 we obtain an almost perfect match.

If the nth term is x^(2n + 1) / (2n + 1)! then we have g(x) = x + x^3 / 3! + x^5 / 5! + x^7 / 7! + … Graphing this over [-3, 3] x [-10, 10] we obtain a graph that appears to be symmetric with respect to the origin. We would expect this of an odd function. It also appears to be a combination of exponentially increasing and decreasing functions. If we compare g(x) with y = sinh x = (e^x – e^-x) / 2 we obtain an almost perfect match.

If the nth term is $(-1)^n \cdot x^{(2n)} / (2n)!$ then we have $f(x) = 1 - x^2 / 2! + x^4 / 4! - x^6 / 6! + x^8 / 8! -$ Graphing this over $[-6, 6] \times [-1, 1]$ we obtain a graph that appears to be symmetric with respect to the y-axis, which we would expect of an even function, and is periodic. If we compare $f(x)$ with $y = \cos x$ we obtain an almost perfect match.

If the nth term is $(-1)^n \cdot x^{(2n + 1)} / (2n + 1)!$ then we have $f(x) = x - x^3 / 3! + x^5 / 5! - x^7 / 7! + \ldots$ Graphing this over $[-6, 6] \times [-1, 1]$ we obtain a graph that appears to be symmetric with respect to the origin, which we would expect of an odd function, and is periodic. If we compare it with $y = \sin x$ we obtain an almost perfect match.

Let's do a similar analysis when the denominator is n instead of n!. If the nth coefficient is $1 / n$ then we have $f(x) = x + x^2 / 2 + x^3 / 3 + x^4 / 4 + x^5 / 5 + \ldots$.Graphing this over $[-1, 1] \times [-1, 4]$ we obtain a graph that appears to be an exponentially increasing function. If we compare it with $y = e^x - 1$ we observe that the curves diverge as x approaches 1. It appears that there may be asymptotic behavior as x approaches 1 but it is not obvious which function this might represent. We will continue with our investigations to see if we can gain further insights that will help us.

If the nth coefficient is $(-1)^{(n - 1)} / n$ then we have $g(x) = x - x^2 / 2 + x^3 / 3 - x^4 / 4 + x^5 / 5 - \ldots$ Graphing this over $[-1, 1] \times [-4, 2]$ we obtain a graph that appears to be a logarithm function shifted 1 unit to the left. If we compare it with $y = \ln (x + 1)$ we obtain a very close match. Notice that

$g(-x) = -x - x^2 / 2 - x^3 / 3 - x^4 / 4 - x^5 / 5 - \ldots = -f(x)$. Therefore $f(x) = - g(-x) = -\ln (-x + 1)$. Graphing this function and series over $[-1, 1] \times [-1, 4]$ confirms that the series $x + x^2 / 2 + x^3 / 3 + x^4 / 4 + x^5 / 5 + \ldots$. represents $f(x) = -\ln (1 - x)$ over $(-1, 1)$.

If the nth term is $x^{(2n)} / (2n)$ then we have $h(x) = x^2 / 2 + x^4 / 4 + x^6 / 6 + x^8 / 8 + \ldots$. Graphing this over $[-1, 1] \times [0, 1]$ we obtain a graph that appears to be symmetric with respect to the y-axis, which we would expect of an even function. Notice that $h(x) = (1/2)f(x^2)$. If we compare the series with $y = -.5\ln (1 - x^2)$ we obtain a close match.

If the nth term is $x^{(2n - 1)} / (2n - 1)$ then we have $q(x) = x + x^3 / 3 + x^5 / 5 + x^7 / 7 + \ldots$. Graphing this over $[-1, 1] \times [-2, 2]$ we obtain a graph that appears to be symmetric with respect to the origin, which we would expect of an odd function. $q(x) = f(x) - h(x) = -\ln (1 - x) + .5\ln(1 - x^2)$. If we compare the series representation of $q(x)$ with $f(x) - h(x) = -\ln (1 - x) + .5\ln (1 - x^2)$ over $[-1, 1] \times [-2, 2]$ we obtain a very close match.

If the nth term is $(-1)^{(n - 1)} \cdot x^{(2n - 1)} / (2n - 1)$ then we have $r(x) = x - x^3 / 3 + x^5 / 5 - x^7 / 7 + ..$ Graphing this over $[-1, 1] \times [-1, 1]$ we obtain a graph that is symmetric with respect to the origin, which we would expect of an odd function, and may have horizontal asymptotes. If we compare $r(x)$ with $y = \arctan x$, we obtain an almost perfect match.

If the nth term is $(-1)^{(n - 1)} \cdot x^{(2n)} / (2n)$ then we have $s(x) = x^2 / 2 - x^4 / 4 + x^6 / 6 - x^8 / 8 + \ldots$ Graphing this over $[-1, 1] \times [0, .5]$ we obtain a graph that is symmetric with respect to the y-axis since $s(x)$ is an even function. $s(x)$ is similar to $h(x) = -.5\ln (1 - x^2)$. If we experiment with the signs, we observe that $s(x)$ is almost a perfect match for $y = .5\ln (1 + x^2)$ over $[-1, 1] \times [0, .5]$.

These explorations require a thorough knowledge of the graphs of the toolkit functions as well as a thorough understanding polynomial algebra. They allow precalculus students to experiment with many different power series and develop an understanding of how an infinite series can represent a function over some interval. They may even obtain results that are often omitted in calculus courses.