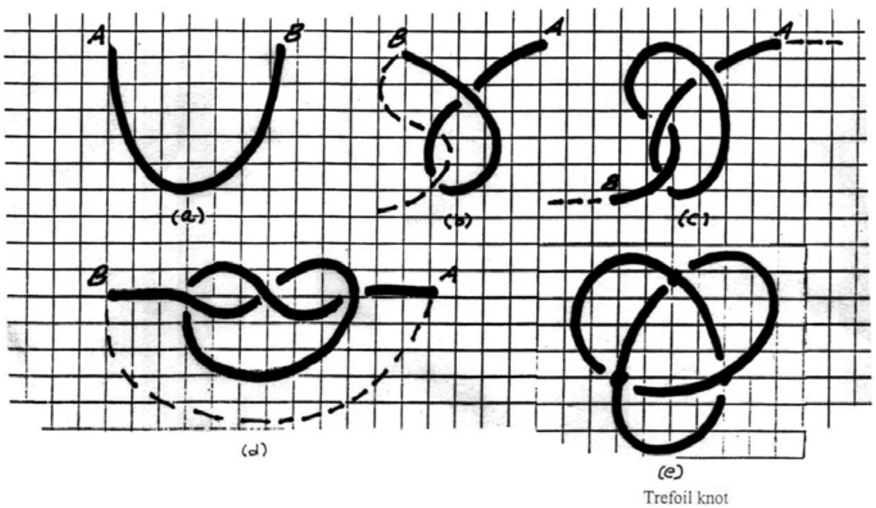
Knots and Soap Film Minimal Surfaces

Nat Friedman
Professor Emeritus Mathematics, University at Albany-SUNY, Albany, NY 12203
artmath@albany.edu

Abstract
We will first give a basic introduction to knots. We will then consider dipping a wire model of a knot in a soap solution and study the corresponding soap film minimal surface (sfms) that forms with the wire knot as boundary. In particular, we will discuss whether the surface is one-sided or two-sided. In most cases the sfms will be one-sided. We will also discuss deforming a knot with a one-sided sfms to obtain a new configuration of the knot that has a two-sided sfms.

The Trefoil Knot
We start by tying a trefoil knot as in Figure 1. Begin with a piece of string with ends A and B as in (a). First cross B over A as in (b). The fact that part B is over part A is indicated by leaving little gaps on each side of part B at the crossing. The dashed line in (b) indicates the next step of inserting B through the loop as shown in (c). We now pull the two ends as indicated by the dashed lines in (c) to obtain (d). The two ends are joined below as indicated by the dashed lines in (d) to obtain the **trefoil knot** in (e). In Italian three is tre and leaf is foglia and the plural leaves is foglii. Since the knot in (e) has three “outer leaves”, the name in Italian is tre foglii and simply trefoil in English.



Tying **A** Trefoil Knot

Figure 1.

To be precise, the knot with loose ends in Figure 1(d) is called an **open knot.** An open knot can always be “untied” since the ends are loose. In order to “lock in” the “knottedness” the ends are joined to form a **closed knot** as in (e). In general, a closed knot is formed by first tying an open knot and then joining the ends. The mathematical theory of knots is concerned with closed knots. For convenience, a closed knot is referred to simply as a knot. For example, the closed knot in Figure 1(e) is the trefoil knot. Actually the picture in Figure 1(e) is a knot **diagram.** Usually the words knot and knot diagram are used interchangeably. A basic question in knot theory is given two knot diagrams, when can one knot be deformed into the other knot? A knot is considered to be made of flexible material and allowable knot deformations are bending, twisting, stretching or shrinking. However, one is not allowed to cut the knot, deform it, and then rejoin the ends. Examples of knot deformations will be discussed below.

Knot Diagrams and Checker Boarding.
A knot diagram can be considered as a map where each enclosed region is a country. The map can be two-colored using shaded and white to distinguish regions that share a boundary. We refer

to the two-coloring as **checker boarding**. It is easy to checker board a knot diagram. For example, consider the trefoil diagram in Figure 2(a).

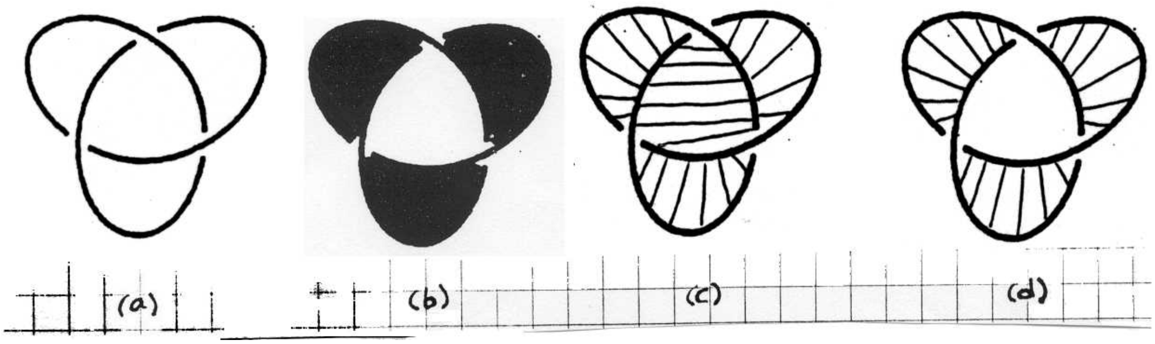


Figure 2.

There are four regions in Figure 2(a). We can first shade the upper left region, as in (b). Since the center region shares its upper left boundary with this shaded region, the center region will be white, as in (b). Since the remaining two outer regions share boundaries with the center white region, they will both be shaded, as in (b). This completes the checker boarding of the diagram in (a). Now suppose we consider a wire model of the trefoil knot and dip it in a soap solution of water, liquid soap, and glycerin. When we remove the wire knot from the soap solution, an initial soap film surface will form on the wire knot, as indicated in (c) by the lines. In particular, there will be a central disk that can be punctured resulting in the final surface indicated in (d), where there is a "space" or "window" in the center. Due to the surface tension of the soap film, this final soap film surface will be stretched so that the surface area is a minimum for all surfaces with the wire knot a boundary. For this reason, we refer to the final soap film surface as a **soap film minimal surface** (sfms). Now note that the sfms corresponds to the shaded regions in (b) and the window corresponds to the white region in (b). Thus the checker boarding predicts the surface (form) and window (space) of the sfms.

We will now consider another example in Figure 3. A knot diagram with four regions is shown in (a). We can begin by shading the upper region, as in (b). Since the two central regions share boundaries with the upper region, they will be white, as in (b). Therefore the lower region will be shaded since it shares boundaries with the white regions. This completes the checker boarding of the diagram in (a). Now suppose a wire model of the knot in (a) is dipped in a soap solution. The initial soap film will appear as in (c). The two central disks can be punctured resulting in the sfms in (d). Note that the check boarding in (b) predicts the surface (form) and windows (space)of the sfms in (d)

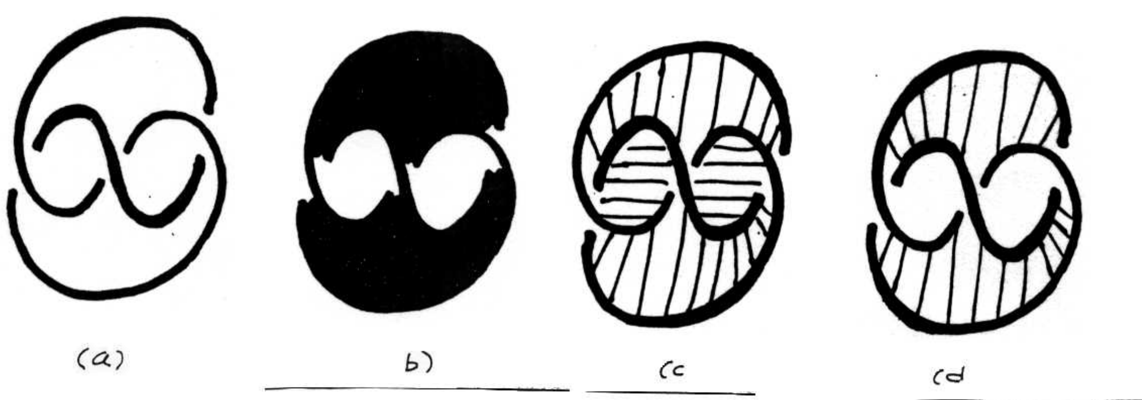


Figure 3.

One-sided and two-sided surfaces.

A Mobius band is an example of a one-sided surface and can be formed as follows. Suppose we start with a narrow rectangular strip of paper that is red on one side and blue on the other side. If point P is chosen on the red side and point Q is chosen on the blue side, then there is no way to get from P to Q moving on the surface without crossing over the edge of the paper. Now give one end a half-twist and then join the two ends together. The result is that the red and blue sides will be joined together. This surface will have only one side, which is partly red and partly blue. It is now possible to get from P to Q without crossing over the edge. This surface is called a Mobius band, named after the German mathematician August Mobius.

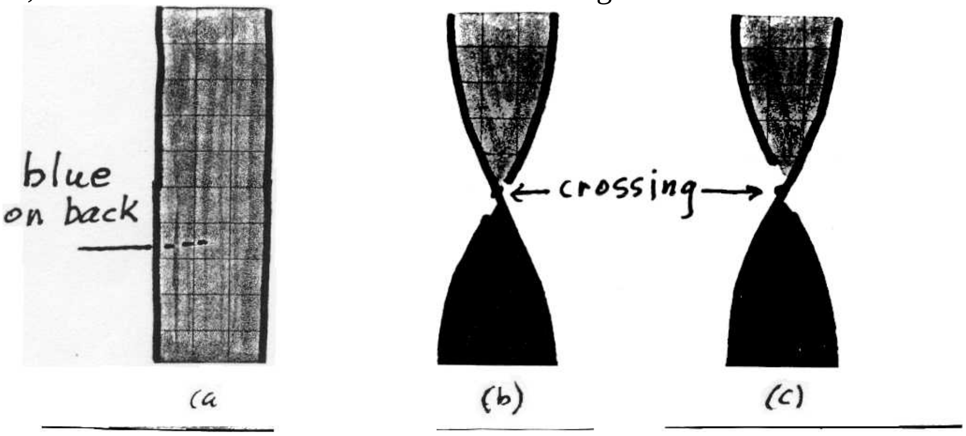


Figure 4.

We will now consider a strip of paper that is red on the front and blue on the back, as in Figure 4(a).

Suppose we give the bottom a half-twist to the right, as in Figure 4(b). The blue side will show up on the bottom. In particular, the colors will switch from red to blue at the crossing due to the half-twist. Similarly, if we give the bottom a half-twist to the left, as in (c), then the colors will switch at the crossing. **Thus a two-sided surface corresponds to the colors switching at a crossing where the surface has a half-twist**. Note that for the back views of the twisted strips in (b) and (c), the colors will be blue at the top and red at the bottom due to the half-twist.

Consider a crossing of a knot diagram, as in Figure 5(a). A sfms corresponding to the knot diagram will have the property that the surface will have a half-twist at each crossing, as in Figure 5(b). If the surface is one–sided, say red, then there is only a red side and the colors do not switch, as in (c). If the surface is two-sided, then the colors do switch, as in (d).

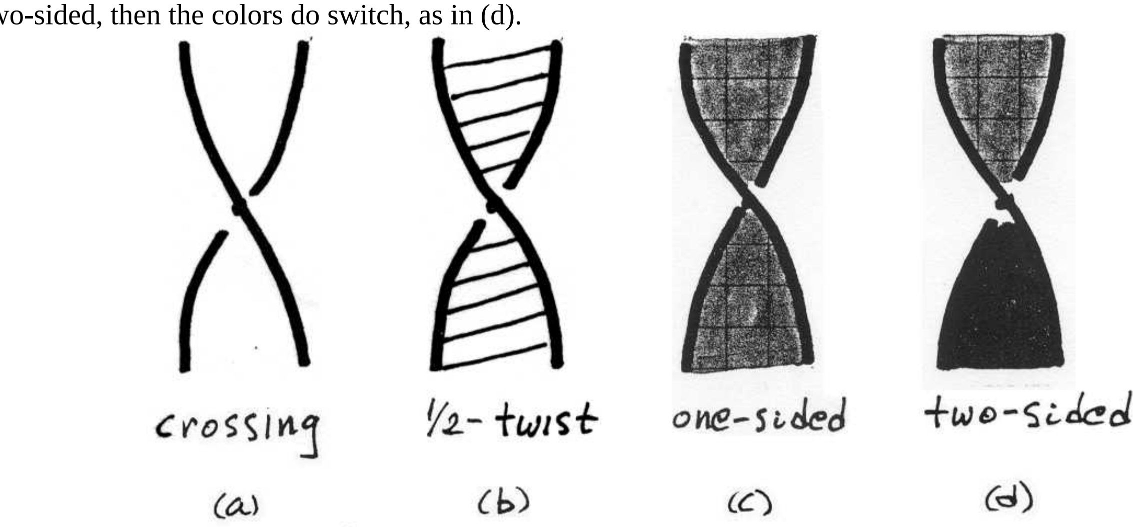


Figure 5.

Now the surface corresponds to the shaded regions of the checker boarding of the knot diagram. **Thus the sfms will be two-sided if the shaded regions can be colored red or blue so that the colors switch at each crossing.** Otherwise the sfms will be one-sided, say all red.

For example, consider the sfms in Figure 3(d), shown in Figure 6(a), where there are only two surface regions. Suppose we color the upper region red (R) and the lower region blue (B), as in (b). The colors are shown in (c). Note that the colors obviously switch at each of the three crossings. Thus the surface is two-sided. If we consider the view in (c) as the front view, then due to the half-twists at each crossing, the surface will be red on the lower region and blue on the upper region in the back view.

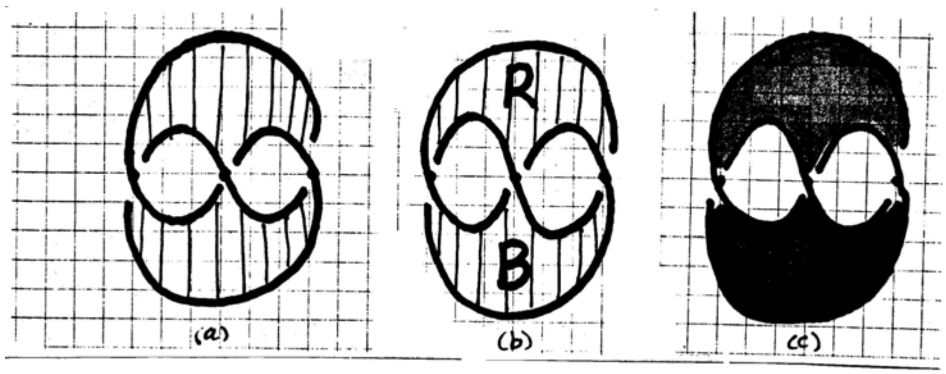


Figure 6.

For another example, consider the sfms in Figure 2(d). If we color the upper left region red, then for two-sided, we would have to color the upper right region blue so the colors switch at the upper crossing. Also the lower region would have to be blue so the colors switch at the lower left crossing. But now the upper right region and the lower region are both blue so the colors do not switch at the lower right crossing. Thus the surface is not two-sided so it must be one-sided. In fact, the surface in Figure 2(d) is a triple-twist Mobius band.

Knot Deformation.

We will now show that the knot corresponding to Figure 2(d) that has a one-sided sfms can be deformed into the knot in Figure 6(a) that has a two-sided sfms. The deformation is shown in Figure 7. The lowest point P in Figure 7(a) is flipped upward as indicated by the dashed lines in (a). The result is shown in (b). Now the points 2 and Q are lowered to obtain the diagram in (c), which is the same as the diagram in Figure 6.

Note that the diagram in Figure 7(a) is labeled trefoil 1, corresponding to a one-sided sfms. The diagram in Figure 7(c) is labeled trefoil 2, corresponding to a two-sided sfms.

Both diagrams are diagrams of the trefoil knot. Thus a knot can have different diagrams. Some are one-sided and some are two-sided.

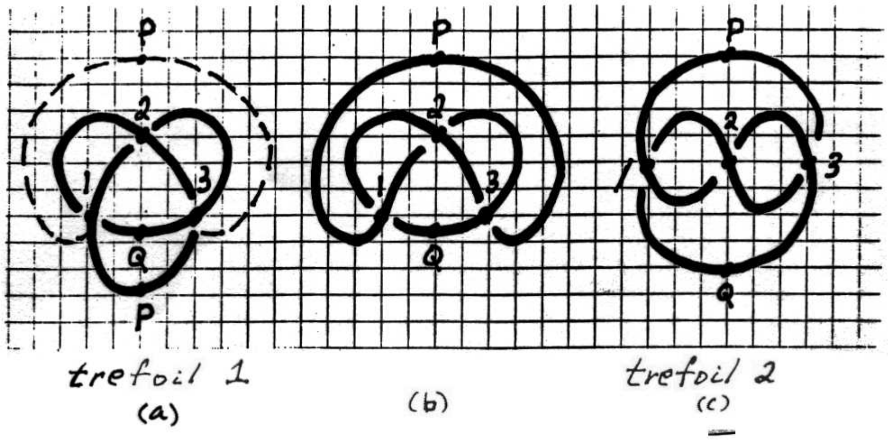


Figure 7.

One more example of deforming a one-sided diagram into a two-sided diagram is shown inn Figure 8. As in the case of the diagram in Figure 2(a), one can check that the diagram in Figure 8(a) has a one-sided sfms. The diagram in Figure 8(e) has a two-sided sfms as indicated. The knot corresponding to the diagram in Figure 8(a) is called the **figure eight knot**. Thus Figure 8(a) is a one-sided diagram for the figure eight knot and Figure 8(c) is a two-sided diagram for the figure 8 knot.

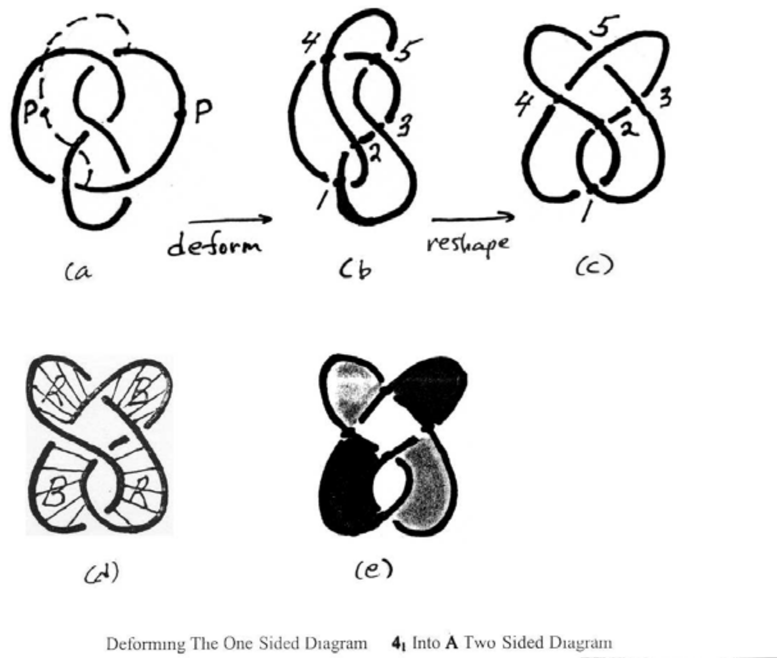


Figure 8.