The Growth of the Dynamic Figural Concept – Sense-Making Strategies Applied to Conceptions of Shape

Doris J. Mohr, PhD, University of Southern Indiana
Crystal Walcott, PhD, Indiana University-Purdue University at Columbus
Signe Kastberg, PhD, Indiana University-Purdue University at Indianapolis
djmohr@usi.edu

Abstract

The view of the discipline of geometry as a rigid subject comprised of inflexible prototypes and mathematical formulas is being replaced by a more dynamic view in which shapes are manipulated using dynamic geometry software and mental imagery. Children's use of prototypes, attributes, and visualization to understand shape has been documented in research. In particular, Fischbein's (1993) view is that each individual's figural concept of a particular shape is unique and is comprised of mental images bound by the formal definition of that shape. Building on Fischbein's ideas, we propose a more dynamic framework informed by children's sense-making strategies that emphasizes not the *formal* definition, but the informal *constructed* definition that each child holds as s/he strives to understand shape. The evolution of this *dynamic figural concept* is based on children's empirical experiences and fits with a more constructivist view of how children learn. Spurred by our curiosity to determine how much of the child's understanding of shape is shared by adults, particularly pre-service teachers, we used our initial framework to categorize a set of responses gathered from two U.S. fourth-grade classrooms and two U.S. pre-service teacher classrooms. This paper will present the development of our framework, the results of our research, as well as implications for teacher education.

Introduction

A good understanding of geometry concepts involves more than the recitation of a verbal definition. Very early in life, children understand shape by identifying examples and non-examples of shape through representing the shape visually. This visual is represented by either drawing a rendering of the shape on paper or by mentally picturing the shape in the mind. As children gain experience with shape, they pay particular attention to certain common attributes of shape as they develop informal definitions of shape. As children continue to gain more and more experience with shape, the verbal definition and the visual image work together to form an evolving understanding of shape.

This past year we analyzed a set of National Assessment of Educational Progress (NAEP) student work samples in order to develop a framework for understanding student sense-making strategies related to shape. In particular, we hoped to inform our future research towards understanding how individuals develop and use the set of images that they associate with a class of geometric figures, along with the attributes of the figures in that class, to sort and classify shapes. Through the examination of the NAEP student work, we developed our research framework, the dynamic figural concept. Using this framework, we analyzed the responses to the same item administered to two samples – one sample of fourth-grade students and one sample of pre-service teachers. The research questions that drove our analysis were, "How do the sense-making strategies utilized by the fourth-graders compare to those used by pre-service teachers when analyzing shapes?" and "How can teacher educators use knowledge of these strategies to inform instruction of pre-service teachers?"

Prior research indicates that individuals utilize visual prototypes that serve as examples and non-examples for classes of shapes. Learners may not only attend to the formal attributes of shape, but their informal or imposed attributes as well. In fact, informal attributes imposed upon visual prototypes often precede formal properties in the development of the conceptual understanding of shape. Even in the presence of a verbal definition, individuals compare shapes using prototypical judgment to assign class membership. The relationship between the visual prototype and the definition of the shape is often disconnected. As in both the mind's eye and in dynamic computer software environments, the attributes of the shape left undefined by the learner result in the ability to freely manipulate the shape by either mental or physical movement of the unfixed properties.

Our Research Framework: The Dynamic Figural Concept

The interpretive framework for this research is derived from an adaptation of Vinner (Vinner & Dreyfus, 1989; Vinner & Hershkowitz, 1980) and Fischbein's (1993) perspectives. We adopt the idea that the understanding of concepts in mathematics, and particularly geometry, involves an invocation of a set of mental images along with a corresponding set of properties for that class of objects. However, our framework is grounded in the understanding of the children's sense-making strategies. We are concerned with the child's view of the mathematical world, and in such, our research lens encompasses the child's process of mathematical understanding through his or her sense-making process. We embrace Fischbein's notion of figural concept through a constructivist philosophy (Noddings, 1990) in which students' actions upon mathematical objects take the forefront. In contrast to Fischbein's view that the formal definition binds the figural concept, we opt to consider a *dynamic figural concept* in which an individual child's figural concept is bound by his or her own *constructed* definition of the object. The term 'dynamic' is essential to the framework we establish. It refers to the child's non-static figural concept at a given place in time, keeping in mind the natural evolution of the child's sense-making strategies as he or she takes in mathematical experiences.

The use of the term dynamic reflects the non-static nature of a child's mathematical world. It also is an indication of the constructivist philosophy that drives our research lens. The dynamic figural concept differs from Fischbein's figural concept in that the structure is easily modified and adjusted based on the child's experienced empirical evidence. The child modifies his or her visual prototype and/or verbal definition for a class of shapes based on changes in the child's view of the mathematical world. These changes are a direct result of experiences gained through the child's sense-making process. As a child constructs new knowledge, he or she calls upon an existing structure of knowledge. New learning experiences cause a refinement or reworking of the structure to incorporate new knowledge gained from the experience. Thus, the dynamic figural concept consists of the visual, verbal, written, symbolic, or formal properties of shape that are valued by an individual child.

In our research we refer to the dynamic figural concept as the construct of images and definitions along with representations other than visual and verbal that a child holds for a particular class of shapes. For example, in thinking about a rectangle in terms of a dynamic figural concept, one has a collection of external representations, such as written symbols, diagrams or graphs, and internal representations (mental images) they associate with rectangle that can be mentally manipulated using an explicit or implicit definition.

Developing Our Framework

A collection of roughly nine hundred student responses to a NAEP item administered in 1992 and 1996 served as the data set for the development of our research framework (see Figure 1). Although the sample is limited to students' written responses, we used it to draw conclusions regarding the students' demonstrated sense-making strategies related to shape classification and the mental models that influence their choice of strategies.

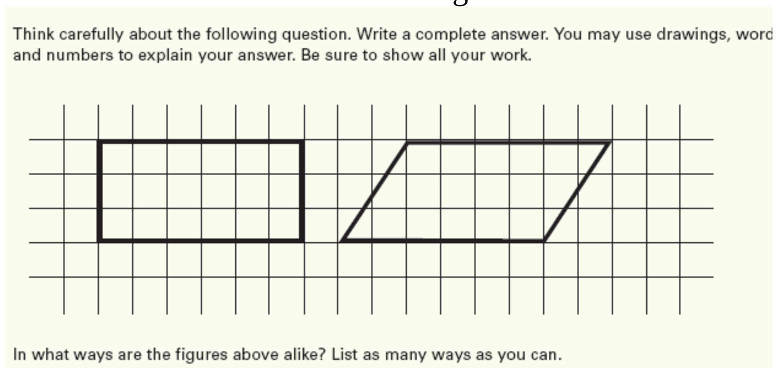

Figure 1

The initial data analysis included several initial attempts to develop categorization schemes for the data. Each member of the research team was given one-third of the data and asked to develop and describe categories. A process of constant comparison (Lincoln & Guba, 1985) best describes this initial stage. Student responses were compared to each other and new categories formed when existing categories were insufficient to classify a piece of student work.

In early classification attempts, we noticed that students attended to the number of sides of the figures. This attention seems to be focused on filtering the objects into a certain class of shapes. Using the number of sides as a filter to establish a class is supported in prior research (Haswgawa, 1997; Lehrer, Jenkins, & Osana, 1998). Students' descriptions of shape in both categories of responses often included an initial reference to common number of sides. This reference to an attribute was so common in the responses that we interpreted these statements as foundational to making a comparison: thus the number of sides act as a filter to establish a class of shapes.

In addition, few students mentioned the parallelogram by name, settling instead for calling the figure a "crooked rectangle" or a "rectangle with leaning sides." Time after time, we read responses referring to the parallelogram as a transformed rectangle. Keeping in mind the dynamic aspect of the figural concept, we decided to separate the data based on responses that either mention or do not mention a 'morphing' (Lehrer et al., 1998) or a mental manipulation on the part of the children of one or both of the shapes. As we set out to interpret the responses that indicate some aspect of morphing we noticed that the children described the manipulation in five distinct manners described in the next section and illustrated in Table 1.

Categories of Morphing

The categories of morphing range from a disjointed view of the figure in terms of its attributes (category 1 morphing) to a holistic view of the figure (category 5 morphing). The first category of morphing, which we deemed *Attribute Morphing*, is an explicit mention of attributes of one of the shapes. These responses included noting that the sides of one figure are bent, slanted, crooked, leaning, or another description of a change of orientation of the sides of one of the figures. The second category of morphing, *One Shape Morphing*, is a description of one of the two shapes being skewed in some manner. Included in these responses were descriptions of one of the figures being slanted, angled, leaning, bent, tilted, or crooked. The first two categories differ in that in the second category the child acknowledges the shape as a whole, whereas in the first the child attends only to the attributes of one shape.

The three remaining categories of morphing responses indicate that the children attended to both shapes. Category three (*Two Shape Morphing*) responses include reference to both figures with a notation that one shape is skewed and the other is not. Category four (*Comparison Morphing*) responses identify the shapes as the same with the added statement that one is skewed. And category five (*Figural Morphing*) responses refer to a description of how one shape can be made the same as the other.

Morphing Categories	Examples of Student Responses
Attribute *Specific mention of attributes or properties of one shape*	The one on the right has lines that are slanted. One has slanted sides and the other doesn't. One has straight sides and one does not.
One shape *Attends to one shape* *Describes skewing of one of the shapes*	The first one is not leaning. One is slanted. One is bent. The one tips to the side.
Two shape *Attends to both shapes* *Describes one shape as skewed, the other as not*	One is leaning. The other one is straight. One is slanty and the other one is straight. The right one is tilted. The left one is straight.
Comparison *Attends to both shapes* *Identifies the shapes as the same* *Describes one shape as skewed*	They are both squares but one is tilted. They are shaped alike. The second one is leaning to the right. They are both the same shape. One is slanted.
Figural *Attends to both shapes* *Describes one shape being made into the other*	If you move the crooked one over, it will be the same. Because if you slide B over it can be the same as A. They bent the first one to get the second one

Table 1

Non-morphing Response Group

In coding the non-morphing response group, we considered all of the remaining student work samples and looked for trends in responses related to attributes and properties. This response group proved to be much less interesting in terms of themes across the work samples; we were unable to link these responses in any meaningful way. The sole commonality across this response group was that students did not mention any manipulation, either physically or mentally, of the two figures.

Methods

Spurred by our curiosity to determine how much of the child's understanding of shape is shared by adults, particularly pre-service teachers, we decided to apply our framework to the analysis of responses to the same NAEP item by fourth-graders and pre-service teachers. Our research team recruited two fourth-grade teachers from a public elementary school to participate in our study. Each of the teachers administered the NAEP item to her respective students. Likewise we recruited two university professors to administer the item to their pre-service elementary teachers enrolled in a math methods or content course. All schools are located in the Midwest region of the United States.

Two researchers worked independently to code the entire sample of responses. Once coded, the researchers met and compared the category assigned to each student work sample. When inconsistencies arose, previously coded NAEP work samples were pulled and used as anchor responses for each morphing category. We focused on evidence of the individual's sense-making strategies instead of the mathematical correctness of the response.

Results

After coding the data, we summarized our findings by type of response (see Table 2). The fourth-graders were more likely to give a response that indicated morphing of some type (59%) compared to the pre-service teachers (32%). Of those responses, the fourth-graders were more likely to view the shapes holistically. Pre-service teachers, on the other hand, more frequently attended to the attributes of the shape, particularly its sides, as being oriented differently. Although based on a small sample, these results seem to indicate that pre-service teachers attend to different aspects of shapes and view them more inflexibly. Children seem to be more comfortable with mentally manipulating the shapes.

RESPONSE CATEGORIES	**Fourth-graders (n = 44)**	**Pre-service teachers (n = 47)**
MORPHING		
Attribute Morphing	7	11
One Shape Morphing	1	0
Two Shape Morphing	10	1
Comparison Morphing	3	1
Figural Morphing	5	2
Total	**26 (59%)**	**15 (32%)**
NON-MORPHING	**18 (41%)**	**32 (68%)**

Table 2

Discussion and Conclusion

The National Council of Teachers of Mathematics (NCTM), by way of their document *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000), has challenged teachers to help in realizing a new vision for school mathematics. Meeting this challenge requires an educational environment that supports teaching and learning that builds upon experience and prior knowledge of the students.

Teachers use classroom discourse, assessment, and observation to make sense of the mathematics of their children in order to properly scaffold their learning. Attempting to understand the child's mathematics becomes futile unless the teacher is able grasp the child's perspective of the task at hand. Identifying the differences in conceptual understandings of adult pre-service teachers and the children they teach informs teacher educators in the preparation of pre-service teachers who are able to grasp the child's perspective.

How do the sense-making strategies utilized by the fourth-graders compare to those used by pre-service teachers when analyzing shapes? In the case of the NAEP item we analyzed, we found marked differences in perspectives displayed in the responses of the pre-service teachers and the fourth-graders. First, many of the fourth-graders viewed the figures holistically while the pre-service teachers attended more to formal properties of the shapes. Secondly, more of the 4th-graders anticipated a physical action on one or both of the figures, or its attributes, than did the pre-service teachers. And finally, the pre-service teachers used more formal language (sometimes incorrectly) to describe the shapes than did the 4th graders. These three categories of issues highlight the marked differences in the ways the two groups understand the classes of rectangle and parallelogram.

How can teacher educators use knowledge of these strategies to inform instruction of pre-service teachers? First, pre-service teachers must be trained in methods to meet students where they are in the development of conceptual understandings. Teacher education programs contribute to the PSSM vision by providing meaningful opportunities for pre-service teachers to build their own strong conceptual understandings and by modeling the use of assessment as a tool to build models of children's understanding of mathematical concepts. Second, by incorporating opportunities for pre-service teachers to view and analyze classroom sets of student work for items such as the one analyzed here, differences between elementary students' and pre-service teachers' uses of and meanings for mathematical terms are discernable. Third, incorporating appropriate tools and technological innovations that encourage the formation of conjecture and proof in pre-service teachers' investigations of geometrical concepts has the potential to shed light on the dynamic nature of the relationship between formal mathematical properties and associated mental imagery.

References

Fishbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*, pp. 139 -162.

Haswgawa, J. (1997). Concept formation of triangles and quadrilaterals in the second grade. *Educational Studies in Mathematics, 32*, 157-179.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study in children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah, NJ: Lawrence Erlbaum Associates.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic Inquiry*. Beverly Hills, CA: Sage Publications.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, Va: Author.

Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 7-18). Reston, VA: National Council of Teachers of Mathematics.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, pp. 356-366.

Vinner, S., & Hershkowitz, R. (1980). *Concept images and common cognitive paths in the development of some simple geometrical concepts.* Paper presented at the Fourth International Conference for the Psychology of Mathematics Education, Berkeley, CA: Lawrence Hall of Science.