8th Grade Students' Understanding of Slope and its Antecedents in a Learning Situation based on Quantitative Reasoning

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Abstract
The graph of the equation y=ax+b is the set of all points (x,y) that satisfy its equation, and this set of points forms a line in the coordinate plane. College students and some high school students may understand this definition. However, based on our experiences with 8th grade students, we realize that such a definition is too sophisticated and abstract for most middle school students. We believe that, for middle school students, a solid understanding of what a line in the coordinate plane represents, lies in the concept of slope and its antecedents. This paper attempts to classify students' understanding of these antecedents of slope based on empirical data collected over a 3-week period in a rural middle school in southeastern USA.

Theoretical Framework

Our theoretical framework is based on Lamon's study of ratio and proportion (1995), which we regard as an antecedent to understanding slope, Lobato's and Siebert's (2002) work on composed units of differences, which we also regard as an antecedent of slope, and Thompson's (1995) explication of quantitative reasoning. According to Lamon, "a ratio is a comparative index that conveys the notion of relative magnitude" (p. 171). Proportion, on the other hand, is about an equivalence of two ratios. Thompson (1995), however, states that "Quantitative reasoning is not reasoning about numbers, it is about reasoning about objects and their measurements (i.e., quantities) and relationships among quantities." (p. 206) All of the above notions are relevant to our work on linear equations and graphs with middle school students where they tried to understand and make sense of the slope concept by relying on their knowledge about antecedents of slope, which were phrases demonstrating an existing relationship between pairs of quantities.

In past research, the notion of slope has been connected with that of "ratio-as-measure" (Lobato, Ellis and Muñoz, 2003). Understanding ratio-as-measure requires proportional reasoning, which entails making sense of the equality of two ratios. According to Lamon (1995), "a student is reasoning proportionally when that student presents valid reasons in support of claims made about the structural relationships that exist when two ratios are equivalent" (pp. 172-173). Slope and its antecedents, are strongly related to such an equivalence. For instance, the "steepness" of a cliff-face with base 300 ft horizontally further out from its peak, and vertical height of 500 ft must be the same, or equivalent to that of a ladder leaning against a wall with its base 12 ft away from the wall and its top touching the wall at a height of 20 ft. In all ratio type examples, there is this coordination of two distinct quantities in a meaningful way. Such a coordination is highly related to a process of operating on these two quantities in order to express a new single quantity. As Lamon (1995) puts it, a ratio is "a complex unit built of two composite units" (p. 171). The processes of unitizing and forming composite units are crucial components in proportional reasoning (Lamon, 1994). Labato and Siebert (2002) introduced the terminology "Composed unit of differences" based on their observation and analysis of students' behavior. A composed unit of differences (CUOD) can be thought of as a single entity resulting from phrases such as "10 centimeters in 4 seconds", "2 ounces every 15 minutes", etc. The phrases in quotation marks may be regarded as an initial stage of conceptualizing what a rate might be. Slope, however, is an artifact of the rectangular coordinate system used to graph such quantitative relations; thus we can call such CUODs antecedents of slope as they are yet to be graphed.

Context and Methodology

This study took place in an 8^{th}-grade classroom in a rural middle school in the southeastern United States. The 24 students were racially and social-economically diverse, with an approximately equal distribution of gender. The first eight class lessons on a unit that focused on graphing and solving systems of linear equations were videotaped using two cameras, one focused on the teacher and the other on the students. This paper focuses on selected problems from Unit 6 of College Preparatory Mathematics, Algebra 1 (CPM, 2002). Some problems were based on contextual situations involving co-varying quantities that could be modeled by a system of two linear equations; others focused on notational form of linear equations (e.g. y-form) and most involved graphing the linear equations to find a solution to the system as well as using algebraic substitution to find a solution.

Each day the classroom video data from the two cameras were viewed and digitally mixed using a picture-in-picture technology. A written summary of the lesson with time-stamps for video reference ("a lesson graph") was created from the mixed video. Subsequently, the corpus of classroom video data was reviewed, along with the related lesson graphs to generate possible themes for a more detailed analysis. A retrospective analysis, using constant comparison methodology, was then undertaken during which the classroom videos were revisited many times in order to generate a thematic analysis from which the following results emerged.

Results

Students supplied various phrases that reflect their interpretation of slope. There was not an exact definition of slope made explicit by any student. By an exact definition, we mean the ratio of the rise over the run for any set of two points on the line itself. However, students were reasoning quantitatively in understanding how the "a" of a given linear equation y=ax+b would be represented by phrases such as "over 1 up 2", "over 1 to the right down 2", etc. Students supplied such phrases in situations where they were plotting their points with or without tables. In the following sections we list the antecedents of slope as interpreted by students, along with transcript data from classroom episodes that support these interpretations. In all transcripts, the teacher is designated by T and students by S#, where each # refers to a different student.

1- Slope as a Direction (The Most General Case): The classroom teacher puts a dot on a transparency of the coordinate axes on the overhead and asks students to guess her line. The following protocol reflects "slope as a direction" spelled out by several students:

S1: Like… which direction is it pointing? [About the line they are supposed to guess]
T: It points straight.
S2: Give us another dot.
T: Why does it make a difference which direction?
S1: [inaudible]
S1: We need another rule.
S3: I think we need another point.
T: Why do you have to have another point?
S4: Give me another direction; one point does not tell you the direction.
T: Why?
S4: I mean… it's just one point… and one point can extend in any direction... well one point does not show the direction…

Students agree they need extra information. Some say they need another point and some need a direction. S4 realizes that there are many lines that can pass through one point. He is also aware that one point and another point on the line is equivalent to one point and a direction. In other words, you need either a set of two points, or just one point and a specific direction in order to draw a line. The latter one is actually very close to the understanding of point-slope formation of a line. The teacher plots another point on the overhead transparency and lets S4 draw the line.

2- Slope as Positive or Negative Increase: Students are asked to graph y=2x+3 and y=2x-1 by first tabulating points. The classroom teacher then asks them to describe their graphs:

T: What else did you notice about the equations that you drew?
S5: They both have the same pattern [about the graphs of y=2x+3 and y=2x-1]
T: Pattern is the same? What do you mean by pattern?
S4: They both increase positively.

In another instant, students graph y=-3x and y=-4x+2:

S4: They both increase negatively.
T: They both increase negatively?
S4: The lines… the lines are…
T: Would you call that an increase?
S4: Decrease maybe...

The phrases by S4 "increase positively" and "increase negatively" can be regarded as an elementary stage of slope as a composed pair of directed quantities: the usage "increase" refers to the change in the y-value, whereas the usage "negatively" refers to the direction in which the x-values are changing for that particular change in the y-values. About the last phrase "decrease maybe" by S4, we can hypothesize that a positive direction of change for the x is already assumed and that "decrease" refers to the second component, namely the change in the y-value. Thus, we conjecture that S4 switched the direction in which he viewed these lines (following the questioning remark from his teacher) from going up and to the left (increasing negatively), to going down and to the right (decreasing positively).

3- Slope Based on Quadrants: The classroom teacher asks students to describe the graphs of the equations y=-2x+5 and y=x-1 without drawing anything yet:

T: I have two equations on the board…Do you know what they might look like when they are graphed?
S4: I think they both are straight lines and the second one will… the line [about y=x-1] will increase positively into positive numbers and the first one [about y=-2x+5] will decrease into negative numbers.
T: So you think y=x-1 will increase and you think y=-2x+5 might go the other way and decrease?
S4: Yes.
S5: y=-2x+5 will start in the second quadrant and go down to the forth quadrant whereas y=x-1 will start from the third quadrant and go up to the first quadrant.

Here S4 is more specific in his use of "increase" and "decrease". By his phrase "increase into positive numbers" he makes a connection to the first quadrant (positive x and positive y values). Similarly, by his second phrase "decrease into negative numbers", he is referring to the forth quadrant. S5, on the other hand, specifies both the initial and the final quadrants for both lines. And these two students do so just by looking at the given equations, before drawing anything. The discussion continues:

T: Does that make sense what he says to all of you? [S6 disagrees]
S6: I think it should start from Quadrant I and go to Quadrant III [about y=-2x+5]
T: y=-2x+5 should start in number 1 and go toward number 3?
S6: Yes.

S5 says what S6 says cannot be possible by giving an example. He chooses x=-5 and obtains a positive number for y, indicating that the line starts in quadrant II.

4- Slope as a Composed Unit of Differences: S5 is the first student to use the phrase "up 2 over 1" when students were asked to describe the graphs of y=2x+3 and y=2x-1:

S5: They both have the same pattern [about the graphs of y=2x+3 and y=2x-1]
T: Pattern is the same? What do you mean by pattern?
S5: They both go up two lines and over one line. ["lines" here refer to the grid lines on the coordinate plane]

In another instant where students were drawing y=2x-5, S5 is again the first student to provide a similar phrase to his "up and over" terminology:

T: Do we have a line now?
S5: It goes up two squares every time..
T: Is that a line?
Students: Yes.
Some students: Line segment.
S5: But we know the pattern..
T: What's the pattern?
S5: It goes over 1 and up 2.
T: Over 1 and up 2 every time?
S5: Yes mam... the previous number plus two equals the next number.

In his first phrase "It goes up two squares every time" S5 is focusing on one element of the composed pair (the increase in the y-value for each unit increase in x) whereas in his second phrase "It goes over 1 and up 2" he focuses on both elements. We can also hypothesize that in his first phrase "It goes up two squares every time", the other element of the composed pair, namely "over 1" (the unit increase in x) is implied in "every time".

In S5's last phrase "the previous number plus two equals the next number" he is referring to the sequence of y-values for each unit increase in the x-value, where the latter is, again, left unstated. Lobato et al (2003) found that many of the high school students in their study created generalizations about slope on the basis of single quantities changing, and that such generalizations constrained their ability to construct slope as a ratio. In S5's case, however, he appears to be reasoning with a composed unit of differences in which one component is left unstated, in some instances, but made explicit in others.

5- Commutative Aspect of Slope as a Composition of Orthogonal Vectors: Students graph y=4x-1 and then supply various versions of composed pairs of directed magnitudes, which lead us to infer a commutative aspect to *our* interpretation of these composed pairs as orthogonal vectors.

T: See if you can find another point for this line.
S7: One, three… [and plots her point on the graph paper on the overhead]
S8: Two, seven… [and plots his point on the graph paper on the overhead]
T: Looks like on the same line to me… What's the pattern S5?
S5: Up 4...
Bo: Up 4 over 1.
S5: Up 4 over 1.
S6: Over 1 up 4.
T: We need to designate a right or left [about the use of "over"]
S3: S6 you said 4x controls what?
S6: Moving your point up or down…

From this exchange, we feel the need to distinguish between expressions "Up 4 over 1" and "Over 1 up 4". They seem to be equivalent, however, what if they were not? An aspect of

CUOD's as components of a vector are evident in this case. We can think of such expressions as horizontal and vertical components for which order does not matter, i.e., orthogonal vectors. S3's question and S6's response suggest that these students are connecting the x-term (along with its numerical parameter) in the algebraic equation with the increase or decrease in the y-values: "4x controls what?" "Moving your point up or down." S6's response, however, is also indicative of the "single quantities changing" generalization that Lobato et al (2003) found in their study.

6- Slope as an Equivalence between Two Composed Units of Differences: S3 suggests using a different CUOD when it comes to graph y=-5x+4.

S3: Ms Jennings, I still believe the 5 has something to do with it…(referring to the 5 in the term -5x)… Alright… look… since we did the graph with 1 and then 5… what if we did the graph by 2's would it be 10?
T: What do you think S9?
S3: I think we should try it...
S9: What do you mean by 2's?
S3: Like we increase [the x values] by 2.
S10: Because for every 2 there is 10 [inaudible]
T: One more time…
S3: I think we increase x by 2... then y would decrease by 10 instead of 5.
T: Do you wanna try that?
S3: The x's… instead of -1, 0, 1… it'll be like -1, 1, 3...
S8: Yeah but it would not really make a difference…
S6: We would just skip a point…
S3: Start with zero… then two… then four… then six…

In the above discussion, S3 used the composed unit of differences (1,5) to define another composed unit of differences (2,10). He was also explicit about the two components of each unit: "we increase x by 2... then y would decrease by 10 instead of 5." This reasoning suggests a proportional variation in these two components, as well as a directional inverse relation (y decreases as x increases). S8's comment that "it would not really make a difference" also indicates that he recognizes the proportional relation between the two CUODs. S6's comment "We would just skip a point" suggests that he may still be focusing on the change in only one of the two components. The above discussion suggests that S3 may be close to constructing slope as an equivalence between two ratios; i.e. a proportion.

7- Slope as a Vector that Generates a Line: When he comes to graph y=3x-4 on the overhead, without making a table, S3 demonstrates, through physical gestures, the possibility of his CUOD taking on the characterization of a vector that he iterates to generate his line. Without doing any calculation, S3 starts by plotting the y-intercept (0,-4) on the overhead transparency:

S3: [pointing to 3 of y=3x-4] since that's the 3 we go up 3 and over 1… [he moves his finger up 3 units and 1 unit to the right and then plots a point at (1,-1)]

He connects these two points with a line segment. He then constructs his line by repeating this "up 3 and over 1" motion. We can see how he counts the squares every time he draws a copy of his line segment. But the difference between this strategy and the other students' strategy is that S3 relies on his pattern whereas the other students in previous class videos relied on their data points in their table (which sometimes were in error and did not produce a straight line). S3's procedure can be regarded as an iteration of a vector (represented by his line segments) to produce his line-graph. He even has a starting point (the y-intercept), a direction that comes from his CUOD, and line segments (the magnitude of each vector) added tip to end. This has the essence of a line being generated via resultant vectors where each resultant vector is the "sum" of the components of the composed pair.

S3: [after plotting 3 more points, i.e., adding his resultant “vectors” tip to end, he says] I know it'll just [keep going—he makes hand gestures that describe how the line extends in both directions]

S10: Mine kinda went the opposite direction.

S1: Mine did too.

S3: Yeah I know it's just… [makes his hand movement again which describes how the line extends in both directions. He also said the line continues although he did not draw it.]

Slope as an iterated vector or CUOD, however, is not so obvious for every student. For instance, S1 says that she does not understand how the “up 3 over 1” strategy works.

Discussion

In a learning environment based on quantitative reasoning, students’ ideas about quantitative relations evolved naturally, without overt intervention by a teacher who taught from a constructivist viewpoint. Students constructed their own interpretations of slope that became more sophisticated as the lessons progressed. This progression can be thought of as the students’ personal historical development of slope – thus, we can call their different interpretations *antecedents of slope.* The closest interpretation to an adult, mathematical definition of slope was S3’s use of an iterated “vector” to produce his line-graph. We believe this is a more mature strategy than simply relying on a CUOD. S3’s method resembles how the parametric equation of a line is generated in a college-level course on analytic geometry. S3's ideas, however, evolved naturally from his initial construction of his CUOD and his later production of proportional CUOD’s. The similarity with a parametric equation for a line is our interpretation from an adult, mathematical perspective.

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