Mathematical Understanding: Analyzing Student Thought Processes while Completing Mathematical Tasks

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Abstract

This paper shares the findings of an exploratory, qualitative investigation of elementary school students' problem solving strategies. Twelve fourth-grade students were given three mathematical tasks about fractions and interviewed during task completion about their problem solving strategies, and their understanding of how to solve the problems. Students were selected to provide variance across their mathematical achievement on the state-wide test and the curricula used in their classroom. While high-achieving students answered more tasks correctly, some explanations included erroneous parts and incorrect mathematical representations. Further, some low-achieving students were able to provide correct and thorough explanations on for tasks that required less cognitive demand. Implications regarding student interviews as both a teaching and research strategy will also be shared.

Background

American students continue to underperform on both international and national assessments of mathematics achievement (National Center for Educational Statistics [NCES], 2000, 2004). Further, successful American students tend to be limited to only successful uses of algorithms and they lack the conceptual understanding of mathematical topics (NCES, 2000).

Recent reforms in mathematics education have recommended that teachers create learning environments that allow students to learn mathematics through experiences with complex problems, which are referred to as mathematical tasks (National Council for Teachers of Mathematics [NCTM], 2000, 2006). Mathematical tasks can vary in terms of their level of cognitive demand. Stein, Grover, & Henningsen (1996) devised a framework of task difficulty that ranges from simple tasks that require students to retrieve factual knowledge to complex, nonalgorithmic tasks that require students to connect mathematical ideas and employ multiple steps en route to completing the tasks. During the implementation of cognitively-demanding mathematical tasks, students have been observed having difficulty identifying problem solving strategies and making sense of how to complete the tasks (Henningsen & Stein, 1997).

While students have been observed solving mathematical tasks, there is little evidence concerning students' thought processes and reasoning while they complete complex mathematical tasks. Nonetheless, teachers have been encouraged to examine students' thought processes as a way of improving their instruction. The literature advocates various instructional approaches such as posing questions (Fennema et al., 1996), facilitating students' discussion of mathematical ideas (Hufferd-Ackles, Fuson, & Sherin, 2004) and examining students' work (Ball & Cohen, 1999). These instructional practices have shown to be beneficial. For example, teachers in the Cognitively Guided Instruction project learned how to design developmentally-appropriate tasks and examine students' thinking. As a result of teachers' instructional strategies, 1st and 2nd grade students' performances on a test of complex word probems increased significantly (Fennema et al., 1996). In another project, fourth-grade students' communication about mathematical ideas included more connections across topics and more elaborate reasoning after teachers began requiring students to explain their reasoning behind their problem solving strategies (Hufferd-Ackles et al., 2004). While encouraging students to share their mathematical thinking benefits students, there is still a lack of empirical research on how students specifically reason complex mathematical tasks.

Research Study

Based on the need to better understand students' mathematical thinking while completing mathematical tasks, this study was driven by the following research questions:

1) What strategies do 4th grade students employ while completing mathematical tasks? Perhaps "Do 4th grade students employ strategies…."
2) How accurate are students' explanations and computations while completing mathematical tasks?

Procedures

Twelve participants were purposefully selected from three elementary schools in a large southeastern city. The schools varied in terms of the mathematics curriculum that they used: reform-based, traditional and a combination of reform-based and traditional materials. Within each school, the four students varied in their performance on the statewide end-of-grade test.

This exploratory study used student interviews to examine students' thoughts and cognitive demands while completing mathematical tasks. In the present study, reserachers posed three grade-level-appropriate tasks about fractions to students (Figure 1). While students were completing each task, they were asked to explain their approach and how they solved each task. Each interview lasted between 8 and 12 minutes and was audiotaped. Students also recorded all of their work on paper.

Figure 1. Tasks given during the interview.

1) $\frac{1}{4} + \frac{2}{4} =$

2) Miguel buys a pack of gum that has eight pieces. He gives two pieces to Tomas, three pieces to Teresa, and one piece to Tristen. How many pieces of gum does Miguel have left? What fraction of the pack does Miguel still have?

3) Tamara and Javon each have pies that are the same size. Tamara cuts her pie into three equal slices and gives one of them away. Javon cuts his pie into four equal slices and gives one of them away. Who has more pie left?

Data Analysis

The inteview data was transcribed into a word processing document. Using an inductive analysis approach (Bogden & Biklen, 2003), interview transcripts were read and coded for frequently occuring themes related to their problem solving strategies and the explanations they used. For each interview transcript, the associated written student work was also analyzed. Each theme was entered in a spreadsheet along with any data related to that theme. After themes were generated, interview transcripts were reread to confirm that the findings accurately depicted the text in each transcript.

Findings

Question 1: What strategies do 4th grade students employ while completing mathematical tasks?

On the first task, 1/4 + 2/4 =, every student solved the task mentally. Eleven students provided an explanation similar to, "1+2 is 3 so the answer is three-fourths." One student answered incorrectly, as he argued “I should add up the top numbers and then add up the bottom numbers so I have 3 over 8.”

Students used two distinct strategies on the second task. Some students worked in a step-by-step approach, where they started with the initial number of sticks of gum (eight) and subtracted as they read the problem. Others added up the number of sticks of gum that were given away and then subtracted from eight. As students completed the task, their methods of finding the answer differed. Some students counted on their fingers, while others wrote every step down on paper. The students who provided the most clear explanation wrote every step on paper and explained why they were subtracting as they completed each computation.

On the third task, students’ strategies included drawing a picture, subtracting fractions using symbolic notation, cross multiplying and finding equivalent fractions. Further, some students simply responded with an answer to the question, “Who had more pie left?” without showing any work or giving any explanation. These students appeared unsure about how to start the task, so they guessed their answer.

Question 2: How accurate are students’ explanations and computations while completing mathematical tasks?

Eleven students accurately answered the first task. However, some students’ explanations included incorrect information. For example, three students from one school explained, “since the bottom number never changes we know that the bottom number has to be four.” In addition, one student explained, “you add up the top numbers and put the answer over the first bottom number.”

At the last two schools interviewed, students were asked if they could think of a story problem that involved adding one-fourth plus two-fourths. Six of the eight students attempted to develop a story problem. Two students created incorrect problems; one represented one-fourth plus one-fourth, while the other represented one plus two and did not involve fractions.

During the second task, nine of the twelve students found the correct answer. Three of those nine students required guidance from the interviewer who asked, “How many pieces did he start with?” and “How did you get the answer?” Without telling, those two questions helped all three students reread the problem and reach the correct answer. The three students who got the correct answer made errors related to operative language; instead of subtracting from eight, they added the numbers in the task.

Students’ explanations for the second task provided evidence that most students were aware of the concept of a fraction as part of a group. Every student correctly identfied the denominator, which they accurately explained was the number of sticks of gum in an entire pack. The errors in the explanations occured when the student had to determine whether to add or subtract the numbers en route to finding the numerator in the answer.

When? For students who found the correct answer had finished, the interviewer asked them, “How much gum had been given away?” Out of the nine students asked, six were able to immediately answer. The other three used paper and pencil to find their answer and were able to eventually find that six-eighths or three-fourths of the pack had been given away.

Eight of the twelve students successfully completed the third task. As stated earlier, students tried a variety of strategies. In some cases, while students’ answers were correct, their explanations were not. The most interesting approach used was cross-multiplying (3 students). After reading the problem, all students correctly determined that they were to compare $\frac{2}{3}$ to $\frac{3}{4}$. Students decided to find the answer by multipling 2x4 and 3x3. After multiplying, students determined that $\frac{3}{4}$ was larger since 9 was larger than 8. Two other students used only the numerator to determine their answer. One reasoned, “That is the correct since three pieces is more than two pieces.” While they arrived at the correct answer, students could not provide an accurate explanation that demonstrated their understanding about how to compare the sizes of fractional parts.

One student could not solve problem 3. She had no strategy for where to begin and she could not tell the researcher what she was thinking as she stared at the problem.

Discussion and Implications

This exploratory study examined the strategies used and explanations given when students completed mathematical tasks. During the early round of data analysis, we have found that while most students were able to successfully complete the three mathematical tasks, their strategies and explanations differed substantially across participants. Numerous students were able to correctly complete the tasks, but gave erroneous explanations related to how they solved the tasks. While asking students to explain their mathematical thinking has been recommended (Fennema et al., 1996; Hufferd-Ackles et al., 2004), more research is needed to determine how to guide struggling students towards correct and deep understandings of the mathematics embedded in the tasks.

Future studies should involve more students to confirm preliminary findings, and more follow-up questions to obtain a clearer sense about students’ mathematical thinking. Further, the use of video to collect data will provide researchers with access to students’ written work and their oral explanations simultaneously. Video would also reveal some very interesting body language exhibited by students as they struggled with solving the problems.

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