Origami, Papierfalten, Papiroflexia: Paper Folding in Mathematics Education

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Abstract
This paper describes the creation, implementation, and assessments of an interdisciplinary course taught at Elon University. Over the last five years, we have successfully tied mathematics, art, history, and culture together in a course called Mathematical Origami. I will share several of the rich problem tasks that generate mathematical discussion in a writing-intensive environment. If you are not fortunate enough to offer an entire course like this, you can certainly use one or more activities to enhance your mathematics curriculum. At the very least, you will walk away with a completed work of origami, papierfalten, or papiroflexia.

Introduction

Tapping into my background of mathematics education and novel approaches to classroom delivery methods, I was asked to create an interdisciplinary course for juniors and seniors from any major investigating mathematical concepts from a unique perspective. Since I have been folding origami for more than 30 years, I readily saw the inherent blend of mathematics, history, culture, and art. The course I created, Mathematical Origami, represents a large set of rich mathematical learning tasks and is now offered on a regular basis at Elon University. We explore mathematical topics from triangle centers, statistics, the concept of function, and a variety of other topics including spirals and basic geometry. These activities require only paper, but tap into rich cultural traditions of not only Japan, but China, Germany, and Spain. We also investigate current advances in technical origami in these countries as well as the United States. This paper will showcase the course's development and implementation focusing on a few specific activities. I will also discuss student motivation, attitudes towards mathematics, and assessment of learning.

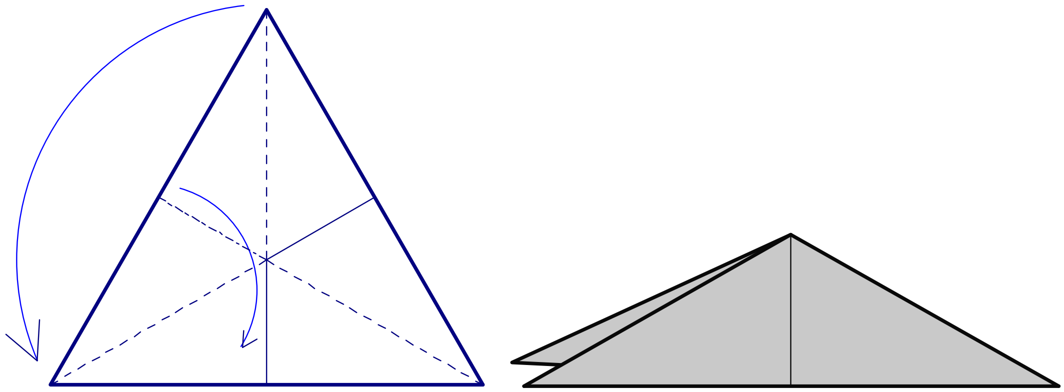
Development

The general studies program at Elon begins with foundation courses in mathematics, writing, wellness, and global studies. Students then branch out into the liberal arts and are asked to complete a capstone experience where they use the tools of many disciplines to study a truly interdisciplinary area. The proposal process ensures that the courses are writing-intensive, interdisciplinary, and foster critical thinking at a high level. Mathematical Origami was my attempt to blend art, history, and culture with my beloved mathematics. Basically I show them how a mathematician and mathematics educator looks at origami and then we investigate paper-folding from the view of an artist, historian, anthropologist, etc. By the end of the course, students use their own major as a starting point for their personal investigation of origami.

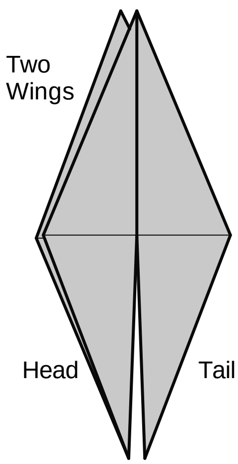
Activities

I prefer a version of mathematics defined by the activities you are exploring rather than predetermined content area. This may stem from my propensity to bring visualization and a traditional geometric approach to most content areas. So, I will share some of the activities we investigate in mathematical origami and let you find the ties to content from geometry to statistics and algebra among others.

One of the first activities in the class involves triangle centers. We discuss ways to create perpendicular bisectors and angle bisectors through paper folding. By folding one vertex to another we can create the perpendicular bisectors of all three sides of the triangle and discuss the properties of the circumcenter. By folding one edge of a triangle onto another, we investigate the point of concurrency among the angle bisectors. Since the incenter always occurs within the triangle, it is not surprising that the incenter is the foundation for a well-known fold in origami, the rabbit-ear.

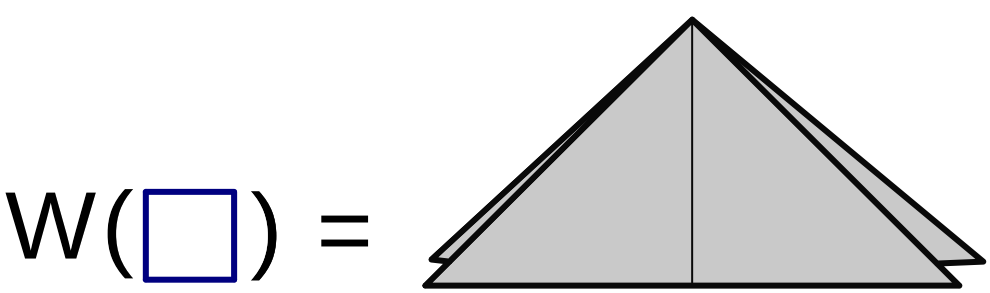


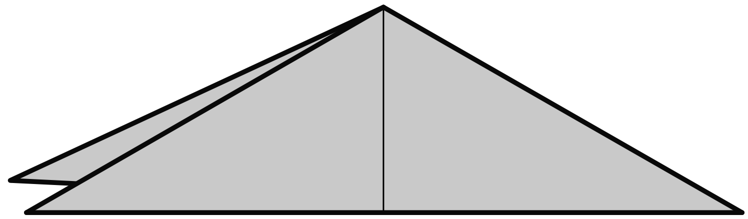
Our second activity draw parallels between origami bases and the concept of function in mathematics. A base is a common way to convey a series of folds in short hand. The bird base, shown below, is a series of approximately eight folds that leaves you with a figure used for many different origami models.



We discuss how the mathematical concept of function can give you a fairly complex set of instructions, but call it by a single name. Suppose the function g(x) takes a number and squares it, then adds the original number back in and subtracts 4. This is written in mathematics as $f(x) = x^2 + x - 4$. I want to know what happens when the number 3 is put inside the function. Well, $f(3) = (3)^2 + (3) - 4 = 9 + 3 - 4 = 12 - 4 = 8$. Again, the final statement $f(3) = 8$ is a quick and abbreviated way to tell you what went in and what came out of the function without having to go through all the detailed steps involved in between.

By thinking of origami as a set of instructions that is applied to a piece of paper, we begin to investigate different inputs (or starting pieces of paper) for our bases. The waterbomb base is a common starting point for many models. If we apply the folding sequence to a square, we get the traditional waterbomb base. If, however, we apply the same folding sequence to a triangle, we get the rabbit ear from the first activity.



W(△) = 

Another activity explores the silver and golden ratios and apply them to paper size. Suppose you wanted a rectangle with the following property: Every time you cut the paper in half, you get two rectangles that are similar in shape to the original. That is, the half is an exact half size copy of the original in scale to the original. This restriction yields the silver rectangle whose length is $\sqrt{2}$ times the width of the paper. This is the basis of the A-system of paper used worldwide. Silver rectangles also make well-balanced airplanes in origami, but we'll get to airplanes shortly.

Now, suppose you wanted a new rectangle with the following property: Every time you cut a square off the end of the sheet, the left over portion is similar in shape to the original. This restriction yields the golden rectangle whose length is $\frac{1+\sqrt{5}}{2}$ times the width. Since most origami models begin with a square of paper, this concept of an inexhaustible supply of squares is quite interesting.

My fourth activity related in this paper focuses on the most under-utilized model in the classroom. A student folding a paper airplane is often used to signify a lack of academic work. In my class, the paper airplanes serve as a vehicle for studying statistics in greater depth. We measure the flight distance of two difficult models and determine if there is a significant difference in their average flight distances. Since Elon University's math requirement has approximately 85% of our students taking statistics in their first year of study, this is a nice way to revisit those notions and apply them.
A fifth activity for mathematical origami focuses on the purely mathematical. I bring in accessible theorems in origami including Maekawa's, Kawasaki's, and Haga's Theorems.

Maekawa's Theorem – When folding, the creases will meet at vertices. In flat-fold origami the difference between the number of Valley folds and Mountain folds at a given vertex will be two. That is, for a model to fold flat, M – V = ± 2.

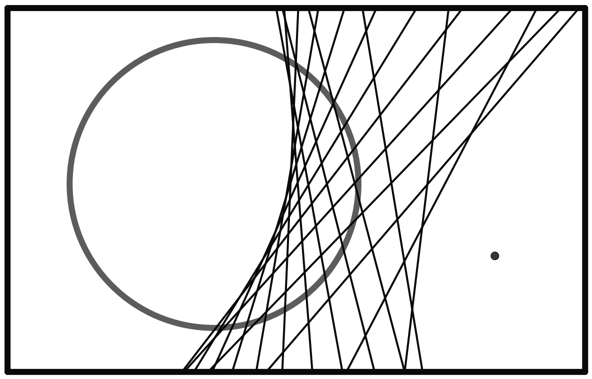
Kawasaki's Theorem – Creases coming together at a vertex define angles around that vertex. In flat-fold origami, alternating angles around the vertex add up to 180^{o}.

Haga's Theorem – Bringing the bottom right corner to the midpoint of the top edge defines a point one-third of the distance along the left-hand side of the sheet.

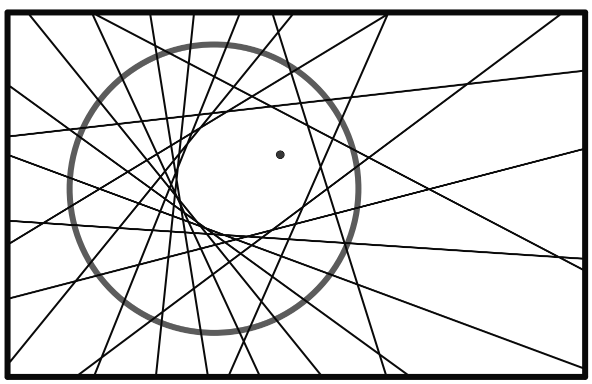
Working through these notions of proof may seem quite difficult for many students, but brings out the satisfaction of proving rather than just theorizing.

While I have quite a large supply of rich mathematical tasks that spring from paper folding, I will only share one more activity in the interests of time and space allowed. The final activity begins with simple folds that create interesting mathematics.

Suppose you fold a given point onto a given circle. This activity works well using waxed paper with the creases showing up as white lines. If the point lies outside the circle, the resulting creases define a hyperbola with foci at the given point and the center of the circle.



If the point lies inside the circle, the resulting creases define an ellipse again with foci at the given point and the center of the circle.



The discussion leading to a proof that the envelope of lines is a hyperbola or an ellipse is the kind of reflection that we want in our classes and reflects the depth of critical thinking you can achieve through paper folding.

Motivation

You may be worried that a course with mathematics in the title may not motivate students outside of the sciences. As the preceding activities have shown, they get the chance to wrestle with deep mathematical notions, but they are also expected to draw connections with origami and their own major.

The course has blossomed into a forum where students from many different disciplines investigate the ties between their field of study and origami. We have had successful projects from disciplines as varied as psychology, biology, and religious studies. Since I am the instructor of the course, the group investigations begin and end with mathematics and mathematics education. Approximately 60% of the way through the course, I turn the class over to individual investigations, thus showing how readily paper folding transfers across disciplines.

Attitudes towards mathematics

Mathematical Origami is one of the most popular interdisciplinary courses offered at Elon University which is no small achievement for a course with mathematics in the title. Student's assessments also point towards strong success for those whose primary learning style is more "hands on". These students report that they learn more mathematics than they thought they could comfortably fit in one semester.

I use the preconception that paper folding is elementary to ease students in and hook them. The same people who balk at my use of the word "function" during the first week of class readily jump in to try proving Haga's Theorem by the middle of the semester. My colleagues and I agree that they are having so much fun that they don't realize how complicated the mathematics has gotten.

Assessment

I must take a moment to discuss the importance of writing in mathematics. I will confess to being a champion of writing across the curriculum since the mid-1990's. I have had my calculus students writing reports and giving presentations for years, but only when asked to create a writing-intensive course did I truly open my eyes to the possibilities of writing to learn mathematics and learning to write mathematically.

My students often complain that they are writing more for my class than they are for typical social science courses. We use writing from daily journal entries which give me the overall progress of the student and the class to quick one-minute papers meant to solidify a specific concept to our 10-page term paper. Each offers me a glimpse into what the students are thinking, how their mathematics is building, and their motivation level.

Conclusion

No matter what you call the art and practice of folding paper, there are some key ideas from mathematics which come alive by investigating folds and creases in paper. Origami, Papier Falten, and Papiroflexia encompass content from geometry to statistics. The students are highly motivated. They report a greater appreciation for how a mathematician views a new area of study. They also report this course being the most interesting from both a general studies point of view as well as their major. For these reasons, I consider paper folding a worthwhile activity in the mathematics classroom.