

## Investigating Properties of Isosceles Trapezoids with the GSP: The case of a Pre-service Teacher

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### Abstract

A four-month pilot teaching-experiment on the learning of geometry was conducted with two pre-service teachers and two in-service teachers. The purpose of the pilot teaching-experiment was to understand how learners, who already have some knowledge of geometric objects, are able to reorganize this knowledge and broaden it in order to develop a better sense for the geometric objects' properties. The guiding principle for the tasks in this teaching-experiment was to use semi-structured constructions using the Geometer's Sketchpad (GSP) and open-ended questions to foster learners' freedom to explore, make conjectures, investigate them, and prove them. The purpose of the task here analyzed was to investigate isosceles trapezoids and their properties and the use of such properties to construct them. In this paper, we analyze the interview in which a pre-service teacher (Susan) works on this task. Susan not only arrived at several of the properties of isosceles trapezoids but also was able to prove them using her knowledge of congruence of triangles.

### Theoretical Rationale

Several studies in the teaching and learning of geometry (Choi-Koh, S. S., 1999; Mariotti, 2000; Jiang, Z., 2002; Christou, Mousoulides, Pittalis, & Pitta-Pantazi 2004; and De Villiers, M., 2004, among others) have used the GSP environment to mediate the interaction between learners and geometric figures as well as the interaction between teacher and learners.

Jiang (2002), in his teaching-experiment, interviewed two pre-service teachers using the GSP environment and he emphasizes that "after conjectures were made and tested, the subjects continued GSP explorations to come up with insight for reasoning and proofs". This teaching-experiment investigated the development of the geometric reasoning of two pre-service teachers as well as the development of their ability to construct proofs. These teachers came to the realization that students could also have a conceptual understanding of geometry if they were given the opportunity to explore geometric figures and make and prove conjectures by themselves rather than to commit to memory the proofs and solutions presented by the teacher.

Another teaching-experiment by Christou, Mousoulides, Pittalis, & Pitta-Pantazi (2004), using prospective primary school teachers, concluded that the GSP was the bridge between exploration and proof in geometry. In this teaching-experiment there was an emphasis in the phase before proof where students constructed a kite using the properties of kites. Two of the three teachers used the property of perpendicularity of its diagonals and the third used the property of two pairs of adjacent equal sides. This knowledge was the starting point in the investigation of the kind of quadrilateral formed by connecting the midpoints of adjacent sides and then proved that this quadrilateral was a rectangle. This teaching-experiment shows that both the GSP and appropriate guiding questions on the part of the teacher can propel students' conceptualizations of conjectures and their proofs.

De Villiers (2004) also using the GSP with prospective secondary-school mathematics teachers asserts that they were able to start from different definitions of isosceles trapezoids and to prove their properties and to construct different isosceles trapezoids according to the corresponding definition. Teachers compared different constructions manifesting their preference for the use of the "axis of symmetry" definition. Then they were also asked to define isosceles trapezoids and

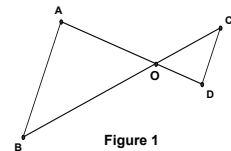
explore their properties. Using the GSP, students were able to construct generalizations as well as specializations dragging vertices or line segments. In the conclusion of the teaching-experiment, the researcher indicates that the use of the GSP and structured activities (at the corresponding Van Hiele level) provide valuable contexts for discoveries, explanations, justifications, and proofs that could be incorporated in the high school geometry curriculum.

It appears as if the mediation of a dynamic software environment, like the GSP, allows students to modify and refine their geometric knowledge while creating a channel for discussion and communication between the teacher and students. Mariotti (2000) indicates that the role of the teacher is fundamental to enhance mathematical discussion and to guide the evolution of student's construction of geometric meanings in a dynamic environment. She also points out that according to her teaching-experiment; the classroom activities in a dynamic environment allowed students to approach theoretical thinking in geometry. Choi-Koh (2001) also investigated a secondary-school student's development as he passed from the first to the last level of the van Hiele's model by means of the GSP environment and open-ended questions on the part of the teacher. The student started at the intuitive level, and then continued to the analytical level, and inductive level, to finally achieve the deductive level. The student used the GSP as a tool not only for verification purposes but also for the construction of counterexamples. According to this researcher, the geometry software provided an interactive environment between the student and his constructions. Such an environment fostered the student's conjectures and the improvement of his geometric thinking as he tried to solve open-ended problems prepared by the teacher. Choi-Koh also emphasizes the need of appropriate tasks for the GSP environment both to link students' intuitions and their ability to analyze geometric objects to generate conjectures and to construct geometric proofs.

## Methodology

*Teaching experiment.* The teaching-experiment methodology fostered long term interactions between researchers and participant teachers to focus on their conceptual constructions and cognitive manifestations. This pilot teaching-experiment lasted one academic semester and consisted of one-to-one interviews with two pre-service teachers and two in-service teachers. One researcher was the interviewer and the other was an active participant observer. The guiding principle for the teaching-experiment was to use semi-structured tasks for the GSP environment as well as open-ended questions to give learners the freedom to explore, to make conjectures, to investigate them, and to prove them. Teachers were interviewed on a weekly basis and each interview lasted 90 minutes. Here we concentrate on the analysis of one interview with Susan (pseudonym), a pre-service secondary teacher.

*Task.* The aim of the task was to help Susan to investigate properties of isosceles trapezoids through a semi-structured figure. Figure 1 was given to her in the GSP environment. In this figure  $ABDC$ . The purpose of the task was to analyze the figure by dragging points and segments and to anticipate what kind of quadrilateral could be constructed from it and how it could be transformed into an isosceles trapezoid. Another purpose of the task was to investigate and prove properties of isosceles trapezoids and how the knowledge of these properties could be used to construct them in the GSP environment.



## Analysis

Susan recognized the obvious triangles OAB and OCD and also a trapezoid if one were to connect the segments AC and BD. She knew that a quadrilateral with one pair of parallel sides was a trapezoid. The following question was about what types of trapezoids she knew, and she answered “isosceles trapezoids”. She also took as well known facts that isosceles trapezoids have the pair of non-parallel sides congruent and that the two pairs of base angles were also congruent and she used these facts, as if they were definitions, to prove new found properties. The proofs were based on the criteria of congruence of triangles and right triangles that we had reviewed at the beginning of the teaching-experiment. Then, the interviewer probed Susan’s knowledge about isosceles trapezoid and challenged her to construct an isosceles trapezoid using the given Figure 1. Her very first reaction was to measure the non-parallel sides AC and BD but immediately she switched into the property of equal diagonals and she measured them. Although Susan described at once two properties of an isosceles trapezoid, she was not able to apply any of them to construct an isosceles trapezoid using Figure 1. She tried to make the given trapezoid an isosceles trapezoid dragging different vertices without complete success. Then Susan constructed an isosceles trapezoid from scratch using the mirror process. She said that she used this way of constructing an isosceles trapezoid in a project she did.

Susan constructed her own figure. She started from two line segments with a common vertex,  $\overline{PR}$  and  $\overline{RM}$  (Figure 2). Then she constructed a perpendicular line on  $\overline{RM}$  at point M and using the GSP she mirrored these two segments over this perpendicular line. She completed the construction of the isosceles trapezoid joining points P and S (see Figure 2). Then she constructed the diagonals and she measured them. Dragging a vertex of the constructed isosceles trapezoid she confirmed her conjecture that the diagonals were always equal (i.e., her figure was robust).

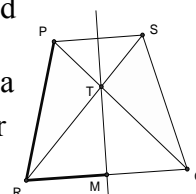


Figure 2

Not being able to construct the isosceles trapezoid using the properties she knew appears to be an indication that her knowledge was only instrumental and not relational in Skemp’s (1987) sense. She pushed herself to remember how she made this construction in her project. When she remembered it, she made the construction using exactly the same process she did in the project. It seemed that she did not know the property of isosceles trapezoids behind this construction (i.e., the bases of isosceles trapezoids have a *common perpendicular bisector* and this perpendicular bisector could be used as a *mirror or axis of symmetry*). If she had known this property well, she could have constructed the “mirror” she needed as the common perpendicular bisector to the bases of the trapezoid into Figure 1 and transform that trapezoid into an isosceles one.

When Susan verified the congruence of the diagonals by dragging any vertex of the isosceles trapezoid in the constructed figure the interviewer pushed her to jump into the following step and prove the congruence of them. Susan expressed her thoughts as follows:

- 45 SUSAN: Ok and I am going to color the triangles PTR and STQ just to color them. So these two triangles triangle PTR and STQ are congruent [*she gives no justification*], ok so that means again I know that the length of RT would be equal to the length of TQ and again the diagonal RS can be RT plus TS and the diagonal TQ can be PT plus TQ. RT is equal to TQ or the length of RT equal to the length of TQ, and the length of ST is equal to PT. So that would make those two diagonals [RS and TQ] congruent. (Figure 2)

The assertion that triangles PTR and STQ were congruent is not valid because the point T (intersection of the diagonals) could or could not belong to the perpendicular bisector (i.e., mirror). Although the question was to prove that the diagonals were congruent, Susan concentrated on the parts formed by the point of their intersection. To justify the congruence of the triangles PTR and STQ she proposed to measure their areas. Susan’s persistence on

measuring as a way of proving indicated a dependence on the software more than on geometric reasoning. Therefore, the interviewer requested a proof independent of measurement. Susan continued the analysis of the figure by considering point M as the midpoint of RQ (the midpoint created by the mirror) and making TM the perpendicular bisector of RQ in order to conclude that TQ was congruent to TR. When the interviewer asked her how she knew that point T (the intersection of diagonals) was on the perpendicular bisector Susan was confused. For that reason the interviewer, through dialogue and with appropriate questions, helped her to understand that she had to compare the triangles PRQ and SQR since their sides PQ and SR were the diagonals of the isosceles trapezoid. She accepted the hint.

88 SUSAN: ok so I am looking at PRQ and SRQ... it's cool. Oh! I get it, the light bulb just went on. All right so  $\Delta PTR$  and  $\Delta STQ$  are congruent by side-angle-side (SAS),  $PR = SQ$ ,  $\angle PRQ = \angle SQR$  [*she took as definition of isosceles trapezoids those trapezoids with congruent base angles*], and  $RQ = RQ$ . Thus, the sides [*diagonals*] PQ and SR are congruent.

Because Susan's first idea was to compare the small triangles PTR and STQ, the interviewer took the opportunity to ask her to prove the congruence of these triangles. Susan considered the congruence of angles RPQ and QSR (as an implication of the congruence of  $\Delta PRQ$  and  $\Delta SRQ$ ), vertical angles PTR and STQ, and sides PR and SQ (because of the other part of her assumed definition of isosceles trapezoids having simultaneously base angles congruent and non-parallel sides congruent as inferred from her mirror-construction of the isosceles trapezoid). This sequence did not fit in ASA and she did not know if it was correct. The instrumental understanding came up one more time. It was clear that she was not able to make the connection with the sum of the angles  $180^\circ$  in order to use the angle-side-angle criterion of congruence. So the interviewer helped her to make this connection. Susan wrote the proof of the congruence of the triangles and its implications  $TR = TQ$  and  $TP = TS$  concluding that the intersection of the diagonals was on the perpendicular bisector of the bases in isosceles trapezoids. Prompted by the interviewer, Susan summarized the properties of isosceles trapezoids.

Then the interviewer went further asking Susan to prove that if in a trapezoid the diagonals are congruent, then the trapezoid is an isosceles trapezoid. It seems that such a proposition was something strange in Susan's mind because since the beginning she also took as definition of isosceles trapezoids those trapezoids with congruent diagonals. She struggled trying to understand what she had to prove and finally she decided to construct a new figure with two equal diagonals to form a quadrilateral. Then she realized that such a quadrilateral was not a trapezoid and she constructed a parallel line and verified that the figure do not have two parallel sides. At that point the interviewer clarified for her that *the given* was that the diagonals in a trapezoid were congruent and therefore she had to prove that under that condition the trapezoid would be isosceles. Thus Susan abandoned the quadrilateral with the equal diagonals and returned to the Figure 3.

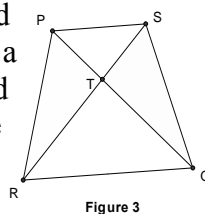


Figure 3

Looking at Figure 3 she started a monologue as she was comparing the big triangles PQR and SRQ and explained to herself that the triangles had only two sides congruent and there was no information about any angle. Susan could not figure out how to prove it and she was repeating that "if the diagonals in a trapezoid are equal then the trapezoid *would* be isosceles". Then the interviewer guided her to figure out that she needed to construct right triangles to overcome the difficulty of the lack of angles. Finally, Susan constructed the appropriate perpendicular lines forming a rectangle and so the right triangles RVS and QUP (Figure 4). After

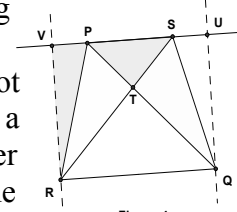


Figure 4

this construction, she proved that  $\Delta RVS$  is congruent to  $\Delta QUP$  using the congruence of the diagonals  $RS=PQ$  and  $RV=UQ$  (perpendicular segments between parallel lines). Then she proved that  $\Delta RVP$  is congruent to  $\Delta QUS$  ( $US=PV$  since  $VP+PS=PS+SU$  and,  $VR=UQ$ , and right triangles). As an implication of this congruence she concluded that  $PR$  is congruent to  $SQ$  and the trapezoid is an isosceles trapezoid.

Then the interviewer gave her a new proposition to prove: “if the intersection of the diagonals lie on the perpendicular bisector of one of the bases of a trapezoid then the trapezoid is an isosceles trapezoid”. Susan constructed the Figure 5 and proved that  $\Delta GEB$  is congruent to  $\Delta GED$  ( $GE=GE$ ,  $BE=DE$  just because it’s the midpoint, and then  $BG=DG$  it’s the hypotenuse of Pythagorean Theorem). Then she proved that  $\Delta AGH$  is congruent to  $\Delta CGH$  [ $\angle AGH = \angle CGH$  because  $\angle AGH = \angle EGD$  vertical angles,

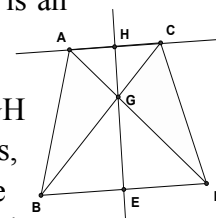


Figure 5

$\angle BGE = \angle CGH$  vertical angles, and  $\angle BGE = \angle DGE$  as an implication of the congruent triangles  $GED$  and  $EGB$ , also  $HG=HG$ , and also the right angles  $GHA$  and  $GHC$  are congruent]. Then, she implied that  $GA=GC$ . Finally, Susan explained that the point  $G$  is equidistant from points  $A$  and  $C$  and therefore it lies on the perpendicular bisector of  $AC$  making the trapezoid an isosceles trapezoid.

When Susan completed the last proof, the interviewer requested a different proof of the prior proposition. Susan used the copy and paste facility and she got Figure 6. “Knowing that point  $G$  lies on the perpendicular bisector of  $BD$  then triangle  $\Delta GBD$  is an isosceles triangle with congruent base angles. Since  $AC$  is parallel to  $BD$  it implies that the alternate interior angles are congruent and therefore  $\Delta GAC$  is isosceles too. Then the sum of the parts of each diagonal are congruent, the diagonals are congruent, and the trapezoid isosceles.” (Susan continues to use the property of the equality of the diagonals as a definition of isosceles trapezoid).

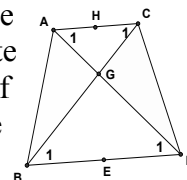


Figure 6

Susan went further and proved that  $\Delta GBA$  and  $\Delta GCD$  ( $AG=GC$ ,  $GD=GB$ , and the vertical angles  $\angle AGB = \angle CGD$ ) are congruent implying that  $AB$  is congruent to  $CD$  and the trapezoid is an isosceles trapezoid. Susan gave the last proofs easily. This fact that she was able to prove a proposition and its converse appears to indicate that Susan have started to make the difference between definition and properties of isosceles trapezoids.

## Conclusion

The GSP provides opportunities to make dynamic constructions and it also fosters ways of thinking and means of exploration that pencil-paper environments do not. It also mediates learners’ dialogues with themselves and with others about the composition and decomposition of geometric figures enhancing, at the same time, the visual modality of their thinking. The analysis of the task presented in this paper indicates that Susan experienced the flexibility of the GSP environment in ways that allowed her to deepen her knowledge about isosceles trapezoids.

The analysis also indicates that, step by step, the pre-service teacher investigated the properties of isosceles trapezoids constructing different figures and exploring them in the GSP environment. Then, she proved the properties using her previous knowledge about the congruence of triangles and the relationships of the angles formed between parallel lines intersected by a transversal. It also indicates that a good understanding of the properties of a geometric figure furnishes the elements to make *robust* constructions in the GSP environment (i.e., a *robust* construction is a construction of a figure which properties remain the same by dragging any of its vertices or sides; that is to say that a *robust* construction generates a whole

family of figures with the same characteristics; this is the meaning of the often used expression “the figure passes the dragging test” when it is a *robust* figure).

In addition, the analysis also indicates that, at the beginning of the interview, Susan had only an instrumental understanding (in Skemp’s sense) of some properties of the isosceles trapezoid because she could not transform the trapezoid in Figure 1 into an isosceles one. Appropriate questioning on the part of the interviewer helped the pre-service teacher to make connections with her previous knowledge and to investigate several properties. Her improvement was also manifested when, at the end of the interview, she gave different proofs of one of the properties.

Susan used six figures, the given and five that she constructed. The GSP also mediated Susan’s explorations, her inferential thinking (inductive and deductive), her self-dialogues, and her constructions as she acted and reacted upon them. It is worthwhile to notice that Susan had different definitions of isosceles trapezoids because she considers their properties as definitions. For example, one was that trapezoids are isosceles trapezoids if they have the non-parallel sides and the pairs of base angles congruent; or those trapezoids which diagonals are equal. Properties of the figure and their converses were unfamiliar to her.

During the four months of the teaching-experiment, the GSP became a useful tool to prepare appropriate tasks and to pose open-ended questions according to the situation. Jones (2000) and Arzarello et al. (2002) agree that the effectiveness of dynamic environment depends not only on carefully designed tasks but also on the teacher’s sensitive guidance and emphasize the software itself does not make an educational change happens but rather it is the teacher who has the important role to guide students’ approach to theoretical thinking in the study of geometry.

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