Increasing Accessibility of Multiplication Facts with Large Factors and Products

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Abstract
Many students have difficulty learning multiplication facts with large factors and products. The curriculum, *Multiplication Matters* (*MM*, Salvo, 2004), was designed to address this issue. *MM* teaches facts grouped by strategies that can make them more accessible to some students. One such strategy is a form of peasant finger multiplication, called Thumbs Up in *MM*. In early teaching with *MM*, the strategy was taught as a finger trick without explanation. In more recent *MM* teachings, attempts have been made to teach the strategy with understanding. This paper contains preliminary results. Further results will be reported at the September 2007 Conference.

Introduction

Multiplication fact fluency serves students in their work on multiplicative structures including multiplication, division, fractions, ratios, and similarity (Behr & Harel, 1990; Vergnaud, 1983). This encompasses a great deal of the mathematics that students do in their school mathematics careers as well as in their lives. Consequently, an important goal of elementary school mathematics instruction is students' development of fluency with basic number combinations for multiplication (National Council of Teachers of Mathematics [NCTM], 2000). This task comes early in the mathematical development of students and can be a defining moment in that development.

Achieving multiplication fact fluency, however, does not appear to be a cognitively trivial task (Ashcraft, 1982), nor does it appear that it can be promoted by unsophisticated educational practices (Baroody, 1985). Many students become frustrated with this task, not because of its mathematical content, but because of its challenge as a verbal exercise (Dehaene, 1997). They associate their frustration, inappropriately, with mathematics. Some students are devastated intellectually and emotionally by their failure to quickly memorize by rote a large number of facts (Ashcraft, Kirk, & Hopko, 1998; Baroody, 1987; Donlan, 1998).

Research on multiplication fact knowledge indicates that traditional curricula and methods for teaching multiplication facts may be flawed or inadequate for some students. Campbell and Graham (1985) report fifth-grade error rates of 17% for the facts 2 x 2 through 9 x 9. It was generally true for their sample that students had been taught facts in the context of times tables. Graham (1987) describes the standard teaching order for simple multiplication problems as starting with small problems and proceeding through the times tables from the twos through the nines. An example of a curriculum using this order is the basal text series, *Holt school mathematics Grade 3* (Nichols et al., 1974).

Empirical Results for Multiplication Facts with Large Factors or Products

A robust finding in empirical studies of multiplication fact knowledge is the *problem size effect*. When students encounter multiplication facts with larger products (> 40), they tend to make more errors than they make on problems with smaller products (≤ 20). LeFevre and Liu (1997), LeFevre et al. (1996), and Clapp (1924) report correlations of error rates with product size. The results of LeFevre et al. are illustrated in Figure 1. While problems with products greater than 40 comprised 17% of the problems in their study, they accounted for 45% of the errors.

Salvo (in press) found similar results on a pretest that she administered. Nine of the 10 most missed problems had products greater than 40 and both factors greater than 5. The nine problems, in order from the most missed, were 8 x 7, 8 x 6, 7 x 9, 6 x 9, 6 x 7, 7 x 7, 9 x 8, 8 x 8, and 9 x 9. They comprised 25% of the test items. They accounted for only 12% of the correct responses but 40% of the errors and omissions.

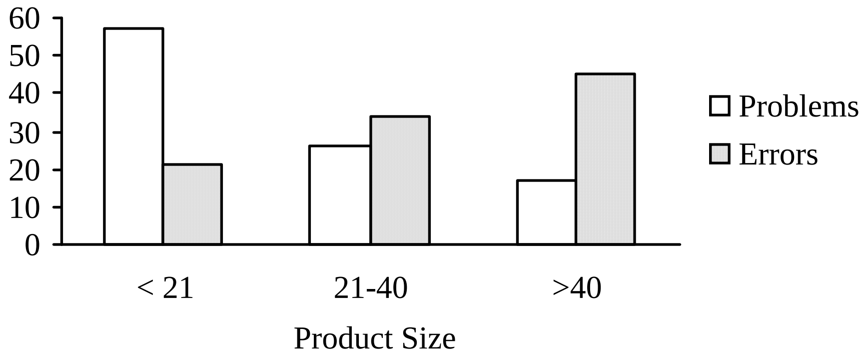

Figure 1. Product size and percent errors. (Data from LeFevre et al., 1996).

Multiplication Fact Strategies

Siegler (1988) describes problem difficulty "as dependent not on problem characteristics as such but rather on how problem characteristics influence ability to execute the alternate strategies that are used on the problem" (p. 263). In other words, it is not the characteristics such as large factor or product size *per se* that make a problem difficult; rather, it is that strategies for problems with such characteristics may be relatively more difficult.

Students, particularly in the acquisition stage, use strategies, and they are flexible, diverse, and variable (Carr & Hettinger, 2003; Hecht, 1999; LeFevre et al., 1996; LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003; Siegler, 1996). Strategies reported for multiplication facts include counting; using recursive strategies such as repeated addition, skip-counting, and deriving products from related known products; using rules for certain factors; using mathematical properties; and using verbal memory aids. Research by Hittmair-Delazer, Semenza, and Denes (1994) suggests that the choice of strategies depends not only on the efficiency of the strategy for a certain problem in a certain context, but also on the strategies available. Students use what they have in a pragmatic way.

Alternative Curricula that Address Facts with Large Products and Factors

Three alternative curricula that address facts with large products and factors are reported in the literature. Each departs from the convention of teaching multiplication facts by tables. They include rearranging the teaching order of the facts, thinking strategies, and strategy group approaches.

Graham (1987) hypothesized that reversing the order of teaching the multiplication tables should improve results on facts with large products. He conducted a study in which he rearranged the learning order of the facts, avoiding multiplication tables, and found that the amount of variance in results accounted for by the structural variables of factor size and product size dropped. The difference was statistically marginal, but suggestive.

In the curriculum called *Look into the Facts* (Thornton & Noxon, 1977), the authors clustered facts by thinking strategies that could be used to solve them. The easier facts were taught first so that they could be used as stepping stones for the more difficult facts; thus the strategies were recursive in nature. Patterns, relationships, finger multiplication for 9s, and commutativity were emphasized. There was heavy emphasis on helping students organize their thinking to create their own or adopt suggested strategies for remembering the facts prior to drill over any given segment of the instruction. Thornton (1978) tested the effectiveness of the approach. Though pretest scores were similar for an experimental and control group, there was a marked difference in the posttest scores of the groups, and the difference was particularly marked for harder facts. The differences were statistically significant.

Salvo (in press) investigated the effects of *MM* (Salvo, 2004), a curriculum that employs a strategy group approach, in which the strategies are largely explicit strategies that can help a student arrive directly at a product, bypassing the use of other known facts as stepping stones to a new product. In *MM*, students are taught the strategies as well the cues they need to recognize to determine which strategy to use. Special effort was made in designing the curriculum to collect, devise, and provide strategies for facts with large factors and products. Facts with larger factors are taught early to provide more exposure to them. Furthermore, they are addressed by finger-enacted strategies that produce readily distinguishable finger and hand shapes. *MM*, like Thornton and Noxon (1977), uses finger multiplication for 9s. Another finger strategy, called Thumbs Up in *MM*, revives an ancient and once widespread method for multiplication in which both factors are 6 through 8 (Dantzig, 1959; Reys et al., 2004).

Salvo (in press) employed a pretest-treatment-posttest design. She looked at correct responses with a focus on problems with large factors and products. Intact classes were taught during their regularly scheduled mathematics sessions during a three-week period. Two groups were taught with *MM* (Salvo, 2004). The *Multiplication Matters* group (*MM*, n = 16) was taught the strategies in *Multiplication Matters,* while the *Multiplication Matters* without Strategies group (*MM* without Strategies, n = 18) was not. The Control group (n = 15) was taught using activities and methods from the *Everyday Mathematics* curriculum (University of Chicago School Mathematics Project, 2001). The principal method taught in *EM* was the array.

Salvo (in press) found no differences in overall gains in correct responses among the groups. However, the *MM* group had greater gains among problems with larger factors and products than the Control group. The *MM* without Strategies and Control groups had greater gains among problems with smaller factors and products than the *MM* group. The findings suggested tailoring multiplication fact teaching methods to the specific facts being taught. While facts with a factor of 9 became accessible through finger multiplication, a strategy students found easy and pleasant, the problems that remained intractable were 6 x 7, 7 x 7, 7 x 8, and 8 x 8. These problems were taught through the strategy called Thumbs Up in *MM*. However, this strategy is fairly complex and few students mastered it. During the study, the strategy was taught but not explained.

Thumbs Up Multiplication Method

Thumbs Up records the results of the binomial multiplication, (10 – a)(10 – b), on fingers. Figure 2 is a pictorial version of a solution for 7 x 8 that begins with a 10 x 10 grid.

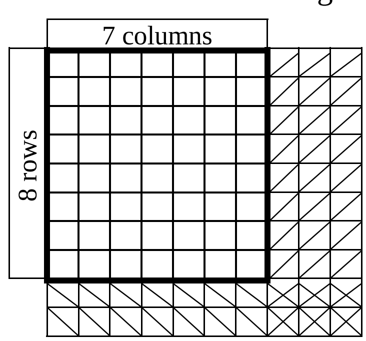

Figure 2. 10 x 10 grid for Thumbs Up explanation.

The complete grid has 100 small squares. By removing three columns of 10 and two rows of 10, the rectangle that remains has dimensions 7 x 8. In that process, the lower right hand corner of dimension 2 x 3 is removed twice.

This can be recorded on the fingers as follows. All 10 fingers are extended to represent 10 10s. Two fingers are bent on one hand to record the removal of two rows. Three fingers are bent on the other hand to record the removal of three columns. That leaves five fingers extended, representing 5 10s, or 50. The bent fingers record the dimensions of the rectangle that is removed twice, 2 x 3. The number of bent fingers on one hand is multiplied by the number of bent fingers on the other hand to get 6. That product is added to the 50 represented by the extended fingers, for a total of 56.

Symbolically, the operation is as follows:

$$\begin{aligned} &(10-2)(10-3) \\ &= (10 \times 10) - (2 \times 10) - (3 \times 10) + (2 \times 3) \\ &= 100 - 20 - 30 + 6 \\ &= 56 \end{aligned}$$

In recent teaching of Thumbs Up in *MM*, the explanation is embodied in a story problem about reforestation. Students are given a picture of a reforestation plot in which there are 10 rows of 10 trees. A fence encloses 7 columns and 8 rows, as in Figure 2. Students are asked to find out how many trees are within the fence by several methods. Students share their answers and explanations. When all are thoroughly satisfied that the answer is 56, the teacher deliberately creates cognitive dissonance by claiming that there are 50 trees in the fenced portion. The teacher explains that there are two columns of 10, or 20, and three rows of 10, or 30, outside the fence. Thus, there must be 50 trees within the fence. At this assertion, several students typically raise their hands in protest and point to the corner that is part of both rows and columns. The twice-removed corner is exactly where student attention must go, as it is essential to understanding Thumbs Up.

Preliminary Results

So far, Thumbs Up has been explained to two groups of children. In the first case, the children were first taught the trick, then the explanation. They seemed confused when the explanation was superimposed on the trick. In the second case, the reforestation problem was used to show children how to record on their fingers what was happening pictorially. A week later, after daily review and practice sessions, the students were observed using Thumbs Up proficiently. Further results will be reported at the September 2007 Conference.

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