The Affects of a Dynamic Program for Geometry on College Students' Understandings of Properties of Quadrilaterals in the Poincaré Disk Model

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Abstract
Prior research on students' uses of technology in the context of Euclidean geometry has suggested technology can be used to support students' understandings of properties of geometrical objects. This study examined the ways in which students used a dynamic geometry tool, *NonEuclid*, as they examined properties of quadrilaterals in the Poincaré Disk model. Five students enrolled in a college geometry course participated in a series of three task-based interviews, one of which focused on quadrilaterals. The Van Hiele levels were used to characterize students' understandings of properties of quadrilaterals and the ways in which students used the technology when reasoning at each of these levels was identified.

Introduction and Theoretical Framework

Traditionally, college geometry is a difficult course for students because it requires them to reason strictly from axioms and postulates rather than informal experiences and intuitive understandings. In order for students to appreciate the importance of the rigorous axiomatic approach, most college geometry courses introduce students to a less intuitive world of non-Euclidean Geometry. Students generally enter a college geometry course with twelve or more years of experience working within the Euclidean system of axioms, and their understandings of figures and relationships within this system are challenged when the axioms are modified. While geometry, in general, is a very visual subject, there are several limitations to students' uses of paper-and-pencil diagrams, especially when it comes to non-Euclidean geometries. A student may create inaccurate misleading diagrams and arrive to incorrect conjectures, or a student may create a correct diagram that is too specific and this may inhibit their ability to derive general conclusions and proofs that go beyond the drawing they have created (Schoenfeld, 1986).

Many mathematics education researchers and professional organizations have suggested the use of dynamic software programs to teach geometry (NCTM, 2000). These software programs enable students to construct accurate diagrams and interact with the diagrams to abstract general properties and relationships, because the ways in which the programs respond to the students' actions is determined by geometrical theorems. Research related to secondary students' uses of such programs has been shown to improve their understandings of geometrical concepts and support their development of formal proofs (Laborde, Kynigos, Hollebrands & Straesser, 2006). Such programs show promise for working with models of hyperbolic geometry, where interpretations of planes, lines, and angles are unconventional. Various technological tools have been developed to assist students in reasoning within different non-Euclidean systems (e.g., *NonEuclid*, Castellanos, 2007), but little research has examined how students' uses of such tools affects their understandings of properties of geometrical figures.

To characterize students' understandings of properties of geometrical figures in particular and geometric thinking in general, a model that is often used is the van Hiele levels (van Hiele, 1986). Central to this model, is the description of five discrete levels of thinking that reflects increasing sophistication in the way in which students are able to reason about geometrical objects and relationships.

Level 1 (*Recognition)*. Students recognize figures holistically. They may recognize a rectangle but not view a rectangle as containing right angles.

Level 2 (*Analysis)*. Students are able to identify relations within a single figure. They may not interrelate figures or properties of figures

Level 3 (*Informal deduction)*. Students can make sense of definitions and are aware of connections among figures. They can logically order the properties of figures by short chains of deductions and understand interrelationships between figures.

Level 4 (*Formal deduction)*. Students are able to construct formal proofs.

Level 5 (*Rigor)*. Students are able to compare and contrast different geometries and axiomatic systems.

Although some researchers (Clements & Battista, 1992) have suggested the existence of a level prior to Level 1 we did not observe reasoning at this level by our students and thus did not include it in the above description.

Prior research examining students' van Hiele levels and their relationship to technology use primarily have been conducted by administering a paper-and-pencil test before and after a technologically-based instructional intervention and these results analyzed to determine whether there is a change in van Hiele level (e.g., Bell, 1998; Johnson, 2003; Moyer, 2003). Results from such studies have been mixed. Several researchers (e.g., Gutierrez, Jaime, & Fortuny, 1991) have suggested using student responses to open-ended items or interviews with students to assess a student's predominate van Hiele level (Fuys, Geddes, & Tishchler, 1988) stating these methods may more accurately report a student's level of geometric thinking. In the current study, interviews with students were conducted as they used technology to complete geometric tasks. A focus was placed on the ways in which technology was used when there was evidence to suggest a student reasoning at a particular van Hiele level.

Methods and Data Sources

For this study, five participants were selected to participate in a series of three interviews conducted by the first two authors of this paper. The third and fourth authors were involved in the teaching of the college geometry course and the development of the interview tasks and protocols. The five participants included two students pursuing bachelors degrees in mathematics: Teresa and Calvin, and three students pursuing degrees in mathematics education: Gail, Ed, and Will. The one-hour interviews were conducted outside of class during the beginning, middle and near the end of the semester during which time they were taking the college geometry course which included five technology-based lab assignments. A video-camera captured students' written work and a video-recording device directly captured students' work on the computer. Interview transcripts were created from the videotapes. Data taken from the second of these three interviews were analyzed for this study.

To characterize college students' understandings of properties of quadrilaterals in the Poincaré disk model and the role of technology, each transcript was analyzed to identify reasoning episodes. Students' geometric thinking within a reasoning episode was coded using the van Hiele levels. To characterize students' uses of technology, codes were created and refined based on the data. These codes included: Drawing, Constructing, Dragging, Diagram, Diagram/Appearance, Diagram/Measures, Diagram/Appearance/Measures, and Dragging Potential. An episode was coded Drawing if a student's goal was to create an object and used the free-hand tools while focusing mainly on the visual appearance of the object on the screen. An episode was coded Constructing if a student's reasoning included a description of properties and features of the technology that were used in the process of creating their geometrical object. In a reasoning episode, if a student made use of the command "Move Point," then the episode was coded Dragging. Episodes coded as Diagram included those such that the drawing on the computer screen was the sole technological element in a student's reasoning. Diagram/Appearance was used to code those episodes where the diagram on the screen and assumptions based on visual aspects of the diagram were used in a student's reasoning. Similarly, Diagram/Measures was used to code episodes where the diagram on the screen and numbers resulting from the use of commands in the "Measurements" menu were used in reasoning. Diagram/Appearance/Measures was used to code reasoning episodes when the diagram on the screen, the appearance of the diagram and the measures taken were all used in a student's reasoning. Episodes coded as Dragging Potential included those where, in the course of their reasoning, students predicted what would occur to a diagram if the diagram, or some portion of the diagram, was dragged.

Results

To facilitate the identification of themes within the data, a table was created to display a count of the number of times the technology was used in a specific way when there was evidence to suggest a student was thinking at a particular van Hiele level (See Table 1).

		Van Hiele Level					
		1	2	3	4	5	Total
Technology Use	Drawing	5	0	0	0	0	5
	Construction	0	5	2	0	0	7
	Drag	0	0	7	0	0	7
	Diagram	0	6	11	6	1	24
	Diagram/Measures	0	40	7	1	0	48
	Diagram/Appearance	0	16	1	0	0	17
	Diagram/Appearance/Measure	0	12	2	0	0	14
	Drag Potential	0	0	4	1	0	5
	Total Technology	5	79	34	8	1	127

Table 1. Ways in which students' used technology while reasoning at particular van Hiele levels

In the process of analyzing students' van Hiele levels and uses of technology, themes related to students' uses of technology and their relationships to the way in which they were reasoning about properties of different quadrilaterals were identified. One theme that became evident is that at level two (Analysis) students mainly made use of the diagram, appearances and measures. A second theme was students' uses of the drag feature while reasoning at level three and four (Informal Deduction and Deduction). A third theme was their use of technology as a visual referent at levels four and five (Deduction and Rigor). Elaboration and examples of these three themes will be provided in the following paragraphs.

Analysis and uses of diagrams, appearances, and measures

During the course of the interviews, students used the diagrams, appearances and measures to reason about a quadrilateral in a variety of ways. 86% (68/79) of instances when students used the diagram on the screen in conjunction with the appearance of the figure and/or measures, students reasoning suggested they were operating at Van Hiele level two (Analysis). In these instances, students often used the technology to identify properties of a quadrilateral and to classify quadrilaterals. In one instance, Calvin used the diagram on the screen and reasoned from the appearance of that diagram to state the diagonals of a rhombus bisected the interior angles. Using the measurement feature of the software, he confirmed his hypothesis. The measurement tool and the ability to create accurate diagrams with the software may support students in identifying relations within a figure, a necessary step in understanding properties of a single geometrical object and a characteristic of geometric thinking at the level of Analysis.

Informal Deduction and Deduction and the use of the drag feature

When students employed the drag feature of the dynamic geometry applet or stated the potential change to the figure if the quadrilateral, or some portion, were dragged, the reasoning students exhibited during these instances suggested they were operating at the third (Informal Deduction) or fourth (Deduction) Van Hiele level. In many of these reasoning instances, the students were either searching for a counterexample to show why a certain property was not true in the Poincaré disk model or attempting to make a generalization for a class of quadrilaterals. For example, Ed used the drag feature to reason that a Saccheri quadrilateral, a quadrilateral with exactly two consecutive right angles and a pair of congruent opposite sides that share a ray of a right angle (See Figure 2a), could not also be a Lambert quadrilateral, a quadrilateral with exactly three right angles (see Figure 1). He dragged a vertex (Vertex *G* in Figure 1) of a Lambert quadrilateral in an attempt to make one pair of opposite sides congruent because one pair of opposite sides of a Saccheri quadrilateral is congruent.

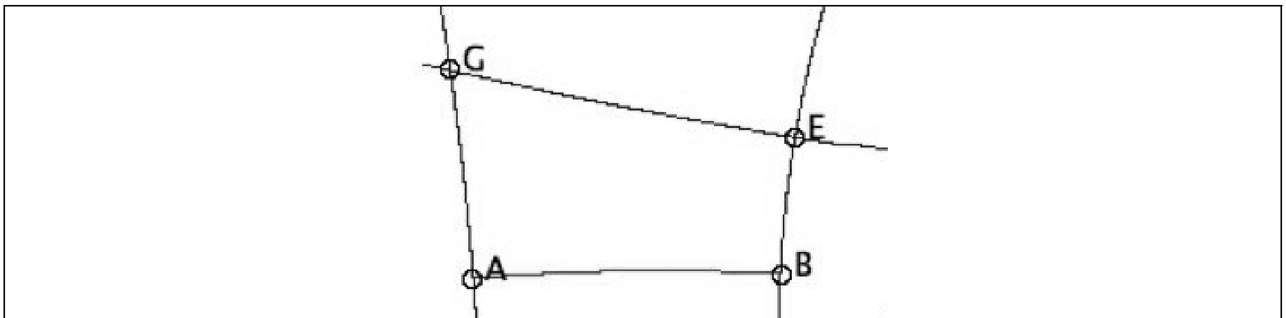

Figure 1. A Lambert Quadrilateral, similar to the one used by Ed and Gail.

By employing this feature he was able to determine that the sides of a Lambert quadrilateral would never be congruent and therefore a Saccheri quadrilateral could not also be a Lambert quadrilateral. Through the use of the drag feature, Ed was able to move from identifying properties (Analysis) to making generalizations (Informal Deduction). The use of the drag feature of a dynamic environment may assist students in making this transition from Analysis to Informal Deduction, but more research is needed to confirm this.

Formal Deduction and Rigor and the use of technology as a visual referent

Instances of technology use occurred at all Van Hiele levels, but only 7% (9/127) of the instances involved students reasoning at the fourth (Formal Deduction) and fifth (Rigor) Van Hiele levels. Of the nine instances where the technology was used when students were operating at the fourth and fifth Van Hiele level, the diagram on the screen was the sole technological use for seven of them. For example, in one instance, Teresa used the Saccheri quadrilateral and its associated properties to prove that an equiangular quadrilateral exists in hyperbolic geometry.

Looking at the diagram of a Saccheri quadrilateral on the screen (Figure 2a), Teresa realized if she reflected the Saccheri quadrilateral about its base ($\overline{AB}$) then the resulting quadrilateral would be equiangular. To explain her reasoning, she used paper and pencil to draw a Saccheri quadrilateral (Figure 2b) and its reflection, marking the summit angles congruent, which she had discovered was a property of the Saccheri quadrilateral in a previous task. In this instance, Teresa's use of the technology was strictly for visualization, but her reasoning suggested that she was operating at the fourth (Formal Deduction) Van Hiele level. In another instance, Gail, was

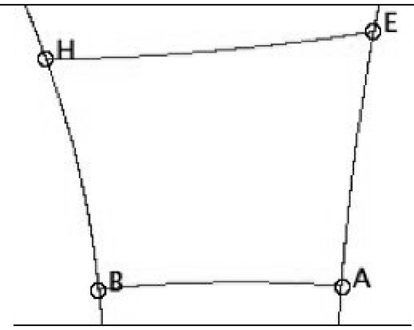

Figure 2a. A Saccheri quadrilateral similar to the one used by Teresa

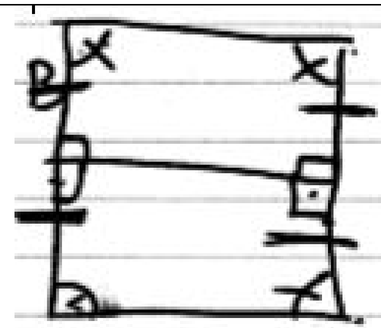
Figure 2b. The equiangular quadrilateral drawn by Teresa.

attempting to determine whether a Saccheri quadrilateral could also be a Lambert quadrilateral. In her reasoning she compared the difference between the sum of the interior angles of quadrilaterals in hyperbolic geometry, less than 360 degrees, and those in Euclidean geometry, equal to 360 degrees. Using the property that the summit angles of a Saccheri quadrilateral are congruent and the definition of a Lambert quadrilateral (Figure 1), she concluded that the Saccheri quadrilateral could not be a Lambert because if it was, then all four interior angles would have to be 90 degrees whose angle sum would be greater than 360 degrees. This contradicts the previously stated property of Hyperbolic geometry which states that the sum of the interior angles of a quadrilateral is less than 360 degrees. In her conclusion she stated, “that the only way for that to occur (for a Saccheri to be a Lambert) is if it were in Euclidean geometry.” Gail was able to compare and reason between two geometric systems, which suggests she was operating at the fifth (Rigor) Van Hiele level. During this reasoning instance, Gail’s sole use of technology was the diagram of a Lambert quadrilateral on the screen.

Discussion

In this study college geometry students appeared to be reasoning at all five van Hiele levels of geometric thinking. While reasoning at these levels, we notice patterns in the ways in students’ uses of technology. For example, at Level One students are mainly using the computer as a drawing tool, relying solely on appearances. At Level Two, the diagram, appearances and measures were used. At Level Three, the drag feature is first employed and at Levels Four and Five the frequency of technological use decreases. A focus on properties of geometrical objects is a characteristic of reasoning at Levels Two and Level Three and we note that it is at these levels where technology is used most. All college geometry students in this study were able to identify right angles, diagonals and sides of individual quadrilaterals and many used the measurement tools to determine how components of a particular quadrilateral were related (e.g., do the diagonals of a parallelogram in the Poincaré disk bisect each other?) Students often used the drag feature to explore class inclusion questions that required thinking at Level Three (e.g., could a Saccheri quadrilateral also be a Lambert quadrilateral?) and referred only to a diagram on the screen when justifying claims or comparing this non-Euclidean geometry to Euclidean geometry. Because measures were used in Level Two and dragging does not occur until Level Three, a question that arises is whether dragging assists students in transitioning from Level Two to Level Three? And while we know the act of dragging alone will not produce advancement in student understanding, what features of the diagram should a student attend to while dragging to facilitate a shift in levels of thinking? We leave these questions for future research.

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