

A New Approach to Conics

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Abstract

In this article we look at a new approach for the teaching of the *conics*, and demonstrate a new method in which *conics* are used as tools in problem solving.

The driving force behind this approach is Geometric Constructions, which creates the opportunity to introduce and develop the very important concept of Geometric Place (Locus) in the early Secondary School years. Through a series of constructions, higher order Loci such as *Conics* are explored, which in return are used as tools in solving more challenging construction problems.

The use of *conics* is a powerful method in problem solving. This becomes evident when dealing with problems with multiple answers and infinite chains of tangent circles.

Examples given will be related to the Parabola but the strategies and methods discussed are also applicable to the *Central Conics*.

Technology is an integral part of this approach and Dynamic Mathematics programs, algebraic and graphics calculators can be used extensively.

Throughout the course, *Euclidean circle geometry* is utilised and in some proofs algebraic methods are shown as alternatives.

1. Introduction

At present, conic sections are generally introduced during the senior levels of high school after students gain sufficient skills in algebra and algebraic manipulation. This is because of the approaches and the definitions used to teach the *conics*. In this article, we propose a new inquiry-based learning approach which allows the introduction of the *conics* at a much earlier stage. It brings with it a large number of investigative projects and problem solving activities. The program starts with geometric constructions geared for plotting and sketching the graphs of conic sections; it provides an informal way of studying conics without the use of algebra. The shapes and some properties of the *conics* such as symmetry are explored. In the second stage, where a formal treatment is used, the equations of the *conics* are obtained and used as tools in a large number of tangency problems related to circles. It allows students to use technology such as graphic and algebraic calculators, and dynamic mathematics programs.

2. Main Features of the New Approach

The main advantages of the new approach to conic sections is that it is computer friendly; the use of technology is an integral part of it.

Its benefits are the following:

- A dynamic computer program can be effectively used to conduct investigations to explore the properties of these important curves through plotting, sketching and the use of transformations.
- Using the *conics* as tools, the newly developed algebraic and graphical methods allow for the wide use of graphics and algebraic calculators in solving a large collection of tangency problems related to circles.
- Presentations can be made in an interesting and lively way through illustrations and animations. Students are encouraged to use interactive animations as they are an excellent means of demonstrating the factors and transformations affecting the shape and form of a conic.

By using an informal approach to *conics*, the proposal allows its introduction as early as Junior High School. At this stage, building geometric constructions related to tangent circles, plotting and sketching of *conics* by free hand or via the aid of a computer, and solving tangency problems by trial and error are just some of the activities that can be conducted.

The concept of limits appears naturally in the study of *Conics* and some of the forms of limits can be introduced in a vivid and meaningful way. In the new approach, the idea of infinitely large and infinitely

small circle (line and point) come up frequently. In high schools usually limits are unfairly neglected and postponed until the need arises in starting Calculus.

The driving force behind this approach is Geometric Constructions, which creates the opportunity to introduce and develop the very important concept of Geometric Place (Locus). Through a series of constructions, higher order Loci such as *Conics* are explored, which in return are used as tools in solving more challenging construction problems.

Problem 1.

A given line L and a circle with radius R centre A are tangent to each other as shown in Figure 1. Construct a circle with radius r to be externally tangent to the circle and also touch the line.

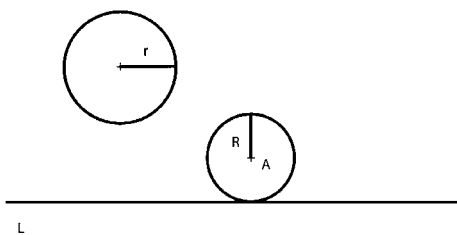


Figure 1

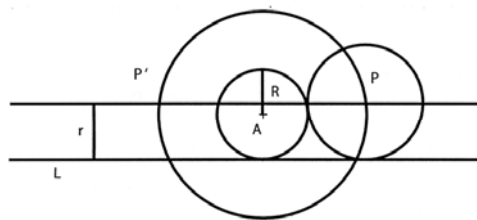


Figure 2

Solution.

Suppose P is the centre of the circle to be constructed (see Figure 2).

(1) P will be on a line parallel to the given line and r units away from it.

(2) P will also be $R + r$ units from A , therefore it will be on a circle with radius $\overline{AP} = R + r$.

If we construct the line and the circle described in (1) and (2), their intersections P and P' are the centres of the two circles forming the two answers to the problem.

Activity 1.

A circle centred at A , radius $R = 2$ cm and a line tangent to the circle are given (see Figure 3). A set of tangent circles are to be constructed tangent to the given circle and line. In Table 1, r shows the radius of the circle to be constructed and $R + r$ represents the distance of its centre to the centre of the given circle.

a. For the values given in the table construct the circles and plot their centres.

b. Sketch the curve that contains the centres of all constructed circles.

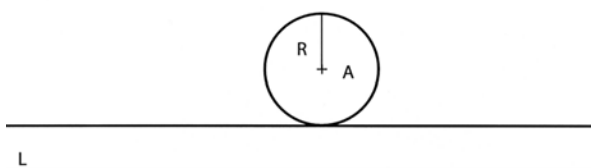


Figure 3

r	0.5	1	1.5	2	2.5	3	0
$R + r$	2.5	3	3.5	4	4.5	5	2

$R = 2$

Table 1

Figure 4 shows the completed task.

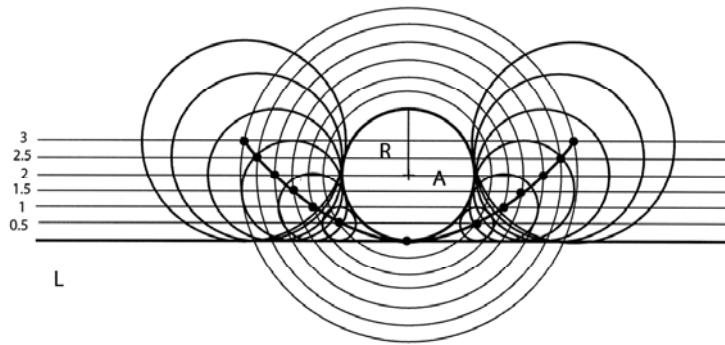


Figure 4

Equation of the Parabola

A circle with centre A , radius R is tangent to a given line L in O . A circle is constructed to be tangent to the given line and circle as shown in Figure 5.

The given line is selected to be the x – axis and the contact point of the given line and circle the Origin.

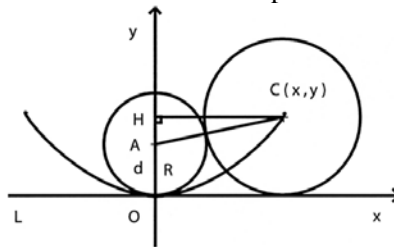


Figure 5

Let the centre of the constructed circle be $P(x, y)$.

From right-angled triangle CAH : $(y + R)^2 = (y - R)^2 + x^2$. It follows that $y = \frac{1}{4R} x^2$.

3. The Method of Conics in Problem Solving

Problem 2.

A line and two congruent circles are mutually tangent. The radius of the circles is R and the contact points of the circles and the line are O and Q (see Figure 6).

Construct a circle to be tangent to the given circles and line.

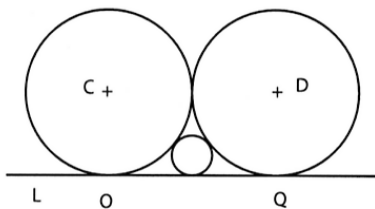


Figure 6

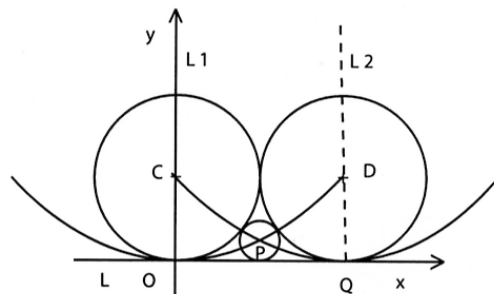


Figure 7

Solution.

Let the given line be the x – axis, and the line through O the Origin (see Figure 7).

If the radius of the given circles is R , the equation of the parabola associated with the circle centred at C and the line is: $y = \frac{x^2}{4R}$.

The equation associated with the other given circle and the line is: $y = \frac{(x-2R)^2}{4R}$.

Solving the two equations simultaneously yields: $x = R$, $y = r = \frac{R}{4}$, where r is the radius of the circle sought.

Extending Problem 2.

Two congruent circles are tangent to the same line and they are on the same side of the line. Construct a circle to be tangent to all of them.

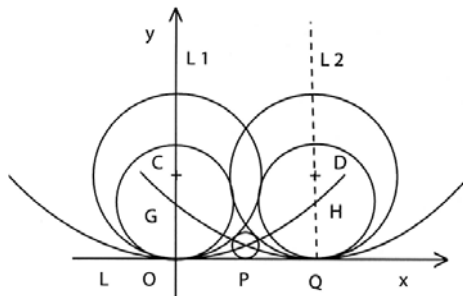


Figure 8

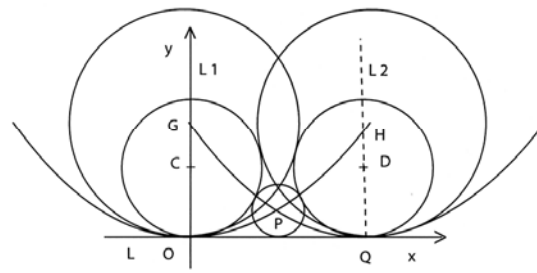


Figure 9

Solution.

Let the distance between the centres of the two given circles be p .

If $p = 2R$, the given circles are tangent

$0 < p < 2R$, the circles overlap

$p > 2R$, they have no common points

The equations of the two parabolas are as follows: $y = \frac{x^2}{4R}$, $y = \frac{(x-p)^2}{4R}$.

The common solution of the two equations gives $x = \frac{p}{2}$.

Using substitution, we get: $y = r = \frac{p^2}{16R}$.

This is the general solution for the three different cases for external tangency.

In Figures 8 and 9, the portion of line L1 above the given line L is the locus of the circle centres which have internal tangency with the circle tangent to L in O, and similarly, the portion of L2 above L is the locus of the centres of all circles having internal tangency with the other given circle in Q.

The intersections of lines L1 and L2 with the two parabolas G and H are the centres of the two circles tangent to one of the given circles internally and the other externally.

Generalizing Problem 2.

After solving the initial problem, by adding a series of tangent circles to come down towards the contact point of one of the circles with the line (see Figure 11), we can create an infinite chain and pose a new problem of a more general nature.

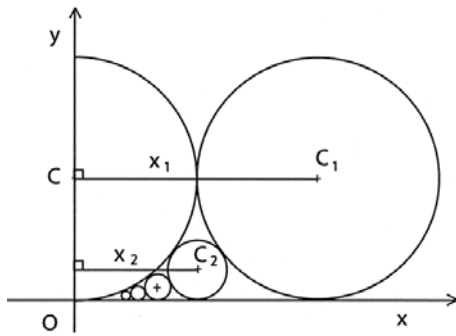


Figure 10

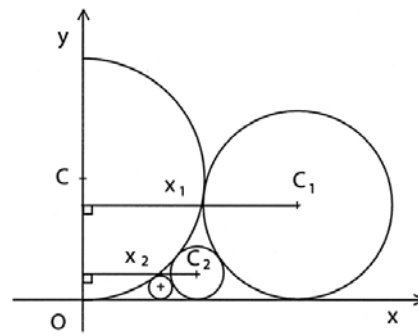


Figure 11

Problem 3.

In the chain shown in Figure 10 find the radius of the n^{th} circle.

Solution.

We know from Pappus' Theorem that: $x_n = 2nr_n$.

Solving this equation simultaneously with the equation of the parabola $y = \frac{x^2}{4R}$ gives $y_n = r_n = \frac{R}{n^2}$.

Extension of Problem 3.

If the chain is formed using two non congruent circles (see Figure 11), where $\frac{x_1}{d_1} = k$, then Pappus' Theorem can be extended to $x_n = (k + n - 1)d_n$.

The common solution of this equation and that of the parabola yields $r_n = \frac{R}{(k + n - 1)^2}$.

4. Extending the Method

We leave the justification of the following theorem to the reader.

Theorem.

The locus of the centres of all circles tangent to a given line and circle is a pair of *confocal* and *coaxial* parabolas opening to the same direction (see Figure 12).

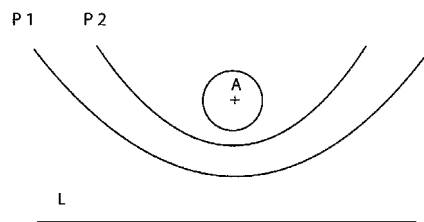


Figure 12

The locus of the centres of all circles externally tangent to the given circle and line is P1, and the locus of the centres of all circles having internal tangency with the given circle also tangent to the given line is P2.

Problem 4.

Given a line and two circles (see Figure 13), construct a circle to be tangent to all of them.

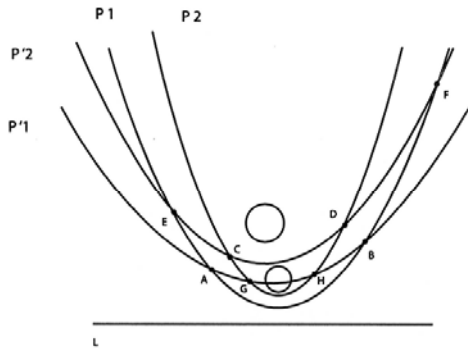


Figure 13

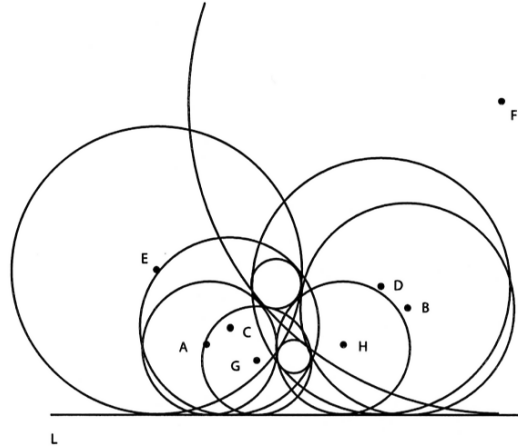


Figure 14

Demonstrative Solution.

In Figure 13, the centres of all circles tangent to the smaller circle are associated with the parabolas labelled as P1 and P2 and the larger one with P'1 and P'2.

The problem has eight solutions and any intersection of two parabolas is the centre of a circle tangent to the given line and two circles. Figure 14 shows all possible solutions.

5. Conclusion

A pre-Calculus, 2 stage program can be designed for the teaching of the important topic of conic sections. In the first informal stage *conics* are plotted, sketched and their properties investigated based on geometric constructions. This is an invaluable experience in forming and developing the concept of *Geometric Loci*.

“Given two things, find the locus of the centres of all circles tangent to both of them” can be an excellent scheme of work to start with. By “a thing” we mean a circle, a line, or a point.

In the second stage the equations of the *conics* are obtained using the *method of coordinates* to investigate their properties further and develop a logical structure leading to the *conics method* used in solving tangency problems related to circles.

The Problem of Apollonius: “Given three things, construct a circle to be tangent to all of them” is an excellent source of short problems with varying degree of difficulty.

In due course, the numerous real life applications of the *conics* can be effectively used to arouse and maintain interest in the *conics*.

Although the examples given in this paper are related to the *Parabola*, the method is also applicable to *central conics* in which ellipses and hyperbolas are used as tools to tackle construction problems. In some cases combinations of two out of the three *conics* can be used.

References

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