The Role of Representations in Growth of Understanding in Pattern-Finding Tasks

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Abstract

The major aim of this paper is to report on a study of the relationship between students' external representations of a mathematical concept and growth of understanding. Using the context of pattern-finding activities, we investigated the relationship between participants' levels of understanding using the Pirie-Kieran Model and of the type of representation used at each level. Our findings suggest that there is an association between participants' levels of understanding and the types of representations used. Further, teachers and researchers might assess students' understandings of a particular mathematical concept through questions or tasks that elicit representing a concept in an alternate form.

Introduction

A central goal of mathematical instruction is for students to formulate and grow in their understandings of mathematical concepts. Some students participate in traditional curricula and instruction that lead them along a convergent path and utilize conventional, external representations. Others interact with reform-oriented curricula, accompanied by complementary pedagogical strategies that encourage the use of nonstandard representations as a means of sense-making. From a constructivist perspective, we assume that students are constantly and actively reorganizing their existing concepts and that this knowledge is relative to them. One does not have direct access to students' conceptions; rather, through the ways that they represent these concepts externally, we may infer the ways in which a student may be thinking about a particular mathematical topic. These representations may take different forms as the students' understandings become more sophisticated. This paper reports on a study of the association between students' representations and the levels of understanding of a particular mathematical topic. Specifically, we address the following research question: What is the role of external representations in students' growth of understanding in pattern-finding tasks?

Review of Relevant Literature

Many theories exist for how individuals learn and grow in their understanding. In addition, the professional and research literature gives much attention to the role of representations in learning and doing mathematics. Some research exists on students' participation in pattern-finding activities. We take the view of learning espoused by von Glasersfeld (1995) for this study. Students construct their own mathematical realities based on their experience, and learning is the changing of these realities as students engage in coordinated actions and operations.

Some researchers have offered theories of how individuals grow in their understanding of mathematics. Sfard (1991) speaks of the dual nature of mathematical conceptions, asserting that a mathematical concept should not be considered fully developed unless the particular process has been reified and can be viewed structurally as well as operationally. Pirie and Kieren (1994) offer another theory that describes an individual's growth of understanding about a specific concept. This model assumes that learning is an iterative, nonlinear process that can be described with eight levels. The knowledge that an individual brings to a new situation is called *primitive knowledge*. The *image making* level consists of creating visual and mental images, while at the *image having* level, one can use these constructs without having to rebuild these notions each time. From these images, one may *notice properties* and *formalize* abstractions of these properties. In *observing*, individuals reflect on and coordinate these abstractions, eventually *structuring* them with rigor. Ultimately, when individuals can pose new questions about these notions, they are *inventising*. One key feature of this model is its nonlinear nature. When an individual reaches a difficulty, they may *fold back* to a previous level of understanding. This previous level of understanding has changed, however, in light of new observations. It is through this folding back that one's knowledge may be enriched and growth of understanding occurs. Though useful in describing different scales of concept formulation, each of these theories posits a beginning phase of coming to understand the new idea and movement towards a noticing of larger characteristics of the concept

Von Glasersfeld (1995) notes that an individual's notion of a concept is inaccessible to all but the individual. Only through representations of that particular concept, specifically those external to the individual, can one infer about another's conception. Representations, then, become crucial to our understanding of how students grow in their mathematical ideas, serving as a mediator in students' growth of understanding and as a means of communicating of that understanding to others. The *Principles and Standards for School Mathematics* (NCTM, 2000) articulates the importance of representations for school mathematics, asserting their necessity in facilitating students' understandings of mathematical concepts and relationships. Research on students' uses of representations indicates that flexibility among these representations is positively related to an increase in conceptual understanding (e.g. Lambertus, 2007). The project for which this study was a pilot has reported that students may use different representations when developing ideas than those that they use to communicate those ideas to their peers (Mojica, Lambertus, Berenson, & Wilson, 2007).

Goldin (2003) offers some fundamental concepts of representations, including the distinction between internal and external and the importance of regarding particular representations as a part of a collection, or a system of representations. It is through these systems that meaning may emerge; "One way of giving meaning to…a given representational system is through their relationship with one another" (p. 277). Cai & Lester (2005) categorized external representations as follows: physical manipulatives, diagrams and graphs, verbal, numerical and tabular, arithmetic symbolic and algebraic symbolic. Because one must infer about conceptions based on the external representations an individual creates, we are focused on external representations and from this point forward, all references to representations refer to those external to the individual.

Exploring students' representations while they are in the process of working on a task gives insight into to their growth in understanding. Smith (2003) asserts the importance of having students work with patterns in context so that they have a chance to create nonstandard representations that describe the actions taking place within the problem situation. By discussing these representations to determine if they are equivalent, students are able to gain a deeper understanding of functions. Lannin (2005) described the ways students typically approach patterning tasks, beginning with a counting strategy, followed by a recursive approach, moving through a stage of using a multiplicative strategy, and ultimately an explicit approach which may be based a context or simply numbers. Thus, pattern activities may provide a useful context for investigating individual's growth of understanding. Another study reports that students' actions when completing a pattern-finding task may be related to their growth of their understanding (Berenson, Mojica, Wilson, Lambertus, & Smith, 2007).

Conceptual Framework

For this study, we selected the Pirie-Kieran Model for Growth of Mathematical Understanding as a way of examining the participants' growth of understanding in a pattern task and Cai & Lester's (2005) explanation of representations as a frame for examining individuals' representations. Several modifications were made. First, we combined the *image making* and *image having* levels into one category. This decision was made because of the difficulty in finding definitive evidence of either level and is not without precedent (cf. Berenson et al., 2007). Another adjustment was the combining of several different categories of representations: manipulatives and diagrams, as well as arithmetic symbolic and algebraic symbolic. Manipulatives were unavailable for some of the tasks (e.g. pentagons) and thus the participants drew the figures. Consequently, we did not feel that we could uniquely classify the use of manipulatives and diagrams. As the participants were university students with ample experience with variables, the use of arithmetic symbolic representations was not anticipated. Finally, as the data was collected through interviews, verbal representations were present at all levels of understanding, likely a result of the interviewer's probes rather than a spontaneous use of a representation. Ultimately, we used the reduced theories depicted in Table 1 to look for a pattern in the representations used at participants' different levels of understanding.

Methods

To investigate our question, we chose an exploratory, multiple case study design (Yin, 1993). This selection was appropriate given that our framework was established before our analysis but the research question emerged after a cursory pass through the data, as the data were collected as a pilot study for another project (cf. Berenson et al. 2007). Further, we made no attempt to attribute causality and merely looked for associational patterns within the data.

Ten university students volunteered to participate in the study, including eight undergraduates enrolled in an introductory psychology course and two graduate students in mathematics education. As a part of our doctoral education, seven novice researchers (two of which are the authors) conducted task-based interviews with the participants. The task involved a series of activities where participants created "pattern block trains" and were asked to predict the perimeters of varying length trains using triangles, squares, and hexagons (see Figure 1). Then, participants were prompted to investigate patterns in the perimeter of trains created with polygons for which there were no manipulatives available, such as pentagons and octagons. Ultimately, the participants were asked to generalize the relationship between the number of arbitrary polygons in a train and the perimeter. The interview protocol was designed by the researchers to elicit instances of folding back to previous levels of understanding.

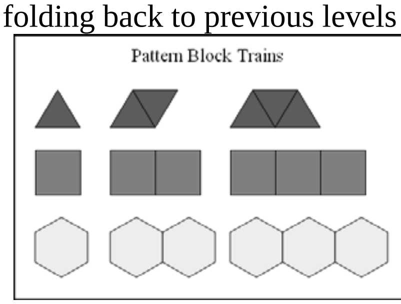

Figure 1. Sample items from the task-based interview. Tasks adapted from Phillips, E. *Patterns and Functions: Addenda Series, Grades 5-8*. Reston, VA: NCTM, 1991.

Verbatim transcripts, videotapes, and students' written work served as the data corpus for our analysis. The transcripts were coded initially for instances of the various levels of understanding from the Pirie-Kieran framework. Next, the transcripts were coded according to the types of representations that the participants were using during each instance. Rather than record the frequency of each instance of the level of understanding and the representation being used, we chose to record the existence of such intersections. This choice was made in part because of the variation present in interviews introduced by the researchers themselves. Moreover, we believed that an increased frequency of occurrences of a specific representation being used did not necessarily indicate a more meaningful usage. For example, a single instance of a participant using a diagram when formalizing may be more significant than many instances of using a diagram while working at image making/image having. From these identified levels of understanding and corresponding representations, we identified times where there was evidence of explicit connections between numeric, visual, and algebraic representations. Participants' written work in concert with video data and the transcripts provided further description of the ways in which the participants were thinking and the representations that they were using.

Findings

Our findings suggest that a participant's choice of representation is associated with their level of understanding. Further, an examination of this association reveals that the participants largely used visual representations to form images of the pattern, but quickly left the visual representations in favor of numeric representations to notice properties. Symbolic representations were used in formalizing and structuring understanding, almost exclusively. Moreover, data on participants' fluency between the three different categories of representations supports this observation.

Using the data summarized in Table 1, we were interested initially in whether there was a statistically significant relationship between the level of understanding of the participant and the type of representation used. The nonparametric Fisher Exact Test was selected due to the small number of observations and is appropriate because of the small number of participants in the study and the reasonable assumption of the independence of cases. In testing the hypothesis of no association between a participant's level of understanding and the type of representation used, we conclude that there is sufficient evidence ($p < .001$) to claim that there is an association between the two. Further examination of Table 1 reveals that this relationship may be interpreted as follows: as a participant grew in their understanding, he or she tended to move from visual representations to numeric ones and then to symbolic representations.

Table 1. Frequency of use of representations at different levels of understanding.

	Visual	Numeric	Symbolic
Image Making & Image Having	100%	40%	-
Property Noticing	90%	100%	-
Formalizing	10%	50%	100%
Structuring	-	20%	60%

Video data and transcriptions provide more information about this association. All participants used blocks or drawn diagrams as they began to create images of the pattern. All used the visual representations to collect data about the pattern in question, but left the context of the blocks as soon as sufficient numerical data was collected. There is evidence that participants used numerical representations at all levels of understanding. These were most often represented in conventional, tabular form.

There is no evidence that symbolic representations were used in more primitive levels of understanding. Whereas images were constructed and properties were noticed based on visual and numeric representations, participants tended to formalize and structure using symbolic representations. All participants formalized with symbolic representations, while only one participant formalized using a visual representation. Seven participants displayed evidence of structuring. Of them, five used only a symbolic representation, one used only a numeric representation, and one used both. When investigating fluency between representations, there was evidence that all participants could translate between their visual and numeric and their numeric and symbolic representations. However, only thirty percent showed evidence of connecting their visual and symbolic representations. This suggests a difficulty in connecting symbolic abstractions to visual, concrete models.

Discussion The purpose of this study was to explore the role of representations in students' growth in understanding in pattern-finding tasks. Pirie and Kieran assert that folding back to a previous level promotes growth of understanding. Our findings suggest that one way to promote folding back is to ask students to re-present their ideas in a different way. Participants who were able to move among representations demonstrated growth in their conceptions of pattern-finding. Several participants removed the context and worked only with numeric and symbolic representations. Presumably, when learners are able to work at an abstract level they have crossed a 'don't need' boundary. That is, they are able to work formally without needing to relate the task to the underlying actions or images. One of the key features of the 'don't need' boundary is that learners can refer back to previous forms of understanding which are embedded and "readily accessible if needed" (Pirie & Kieran, 1994, p. 172). Yet, the students in our study who were quick to use symbolic representations were often unable to re-present using visual representations, suggesting that these previous forms of understanding were not readily accessible to them. Asking participants to re-present opened a window into a lack of understanding of the entire concept. Prompting participants to fold back with re-presenting not only promoted growth in their understanding of pattern-finding, but in some instances revealed a lack of understanding. Thus, proficiency with one representation does not imply understanding. Our data reinforce Goldin's (2003) emphasis on the importance of recognizing a system of representations instead of considering representations in isolation. Results suggest that one way teachers may be able to help students grow in their understanding of pattern-finding tasks is by encouraging them to fold back by employing multiple representations.

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