Proof in Dynamic Geometry: More than Verification

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Abstract
This paper will discuss some important functions of proof, other than the traditional one of verification, which can be used to make proof more meaningful in a dynamic geometry context. For example, it's argued that proof should first be introduced as a means of explanation, and then later on understanding of other functions of proof such as discovery, verification, systematization, etc. can be developed.

Introduction

Traditionally, proof in the geometry classroom has been presented only as a means of obtaining certainty; i.e. to try and create doubts about the validity of one's empirical observations, and thereby attempting to motivate a need for deductive proof. This approach stems largely from a narrow formalist view that the only function of proof is the verification of the correctness of mathematical statements.

However, proof has many other important functions within mathematics, which in some situations are of far greater importance than that of mere verification. Some of these are (compare De Villiers, 2003):

- *explanation* (providing insight into why it is true)
- *discovery* (the discovery or invention of new results)
- *intellectual challenge* (the self-realization/fulfilment derived from constructing a proof)
- *systematisation* (the organisation of various results into a deductive system of axioms, concepts and theorems)

Proof as a means of explanation

With very few exceptions, mathematics teachers seem to believe that only proof provides certainty for the mathematician and that it is therefore the only authority for establishing the validity of a conjecture. However, proof is not necessarily a prerequisite for conviction—to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof.

The well-known George Polya (1954:83-84) writes:

> "*... having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong* ***confidence*** *in the theorem. Without such* ***confidence*** *we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is* ***true****, you start* ***proving*** *it.*" (bold added)

In situations like the above where conviction prior to proof provides the motivation for a proof, the function of the proof clearly must be something other than verification/conviction.

In real mathematical research, personal conviction usually depends on a combination of intuition, quasi-empirical verification and the existence of a logical (but not necessarily rigourous) proof. In fact, a very high level of conviction may sometimes be reached even in the absence of a proof.

Some Examples

Most teachers have probably observed the frustration that children experience when required to prove (verify) an intuitively self-evident result, such as the *pons asinorum* (i.e. the base angles of an isosceles triangle are equal). Suppose children are now guided, within a dynamic geometry environment such as *Cabri* or *Sketchpad,* to construct an isosceles triangle by reflection around line AD as shown in Figure 1, and then to measure the base angles. Since they can quickly and accurately check many cases by dragging, the experience is generally overwhelmingly convincing.

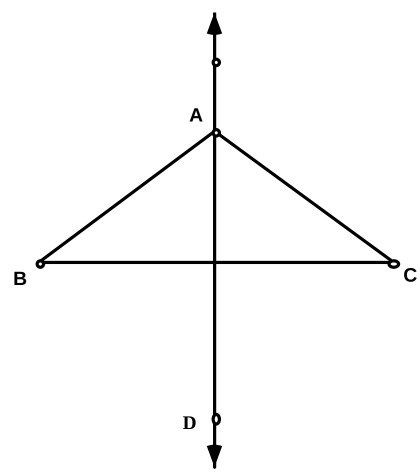

Figure 1

To now try and convince novices of the need to further "*make sure*" of the truth of this result is not only futile, but in my opinion also a poor representation of genuine mathematical practice. The issue at stake here is not the removal of doubt at all, but rather to explain why it is true.

Generally, plane geometric concepts such as triangles and quadrilaterals should in my view as far as possible be introduced as above via symmetry transformations. The reason is that the properties subsequently discovered are then very easily explained, and there is no need to go into relatively complicated congruency proofs at this stage. Also recommended at this stage is to completely try and avoid using the word "*proof*" and instead to ask learners for a "*logical explanation*". For example, something like the following should be quite satisfactory: sides *AB* and *AC*, angles *ABC* and *ACB* are all equal, because folding (reflecting) the triangle around the constructed axis of symmetry through A (by definition of symmetry) maps *AB* onto *AC*, and also angle *ABC* onto angle *ACB*.

In De Villiers (2003), a kite is similarly introduced by reflection around its axis of symmetry through a pair of opposite vertices and its properties subsequently explored and logically explained in terms of its axis of symmetry. In exactly the same way, an isosceles trapezium and rectangle can be introduced (and their properties explained) via reflection in their axes of symmetry through opposite sides, a parallelogram through a half-turn of a triangle through the midpoint of one of its sides, etc.

It is also quite in order at the introductory level of proof in geometry to simply assume some basic geometry results as "*facts*", and then using them to logically explain other more interesting and exciting results. This is quite acceptable from the modern axiomatic view that axioms are necessary assumptions rather than “self-evident” truths.

So for example, one could conveniently assume that the segment connecting the midpoints of two sides of a triangle is parallel to the third side, in order to logically explain a result that children find far more intriguing, namely, Varignon's theorem (i.e. the midpoints of the sides of any quadrilateral form a parallelogram)! One can always come back some time later to these assumptions when the students have matured a little and at least begun to understand and appreciate the systematization function of proof.

In fact, historically, much of geometry did not start by working logically forward from explicitly stated axioms and elementary results to more complicated results, but generally backwards from them, towards identifying the underlying assumptions and axioms.

Note that although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of empirical verification by hand or computer, this generally provides no satisfactory explanation why the conjecture may be true. It merely confirms that it is true, and even though considering more and more examples will increase one's confidence, it gives no psychological satisfactory sense of illumination - no insight or understanding into how or why the conjecture is the consequence of other familiar results.

A significant finding of a study by Mudaly & De Villiers (2000) was that young novices appear to display a need for an explanation (deeper understanding) of Viviani's theorem, quite independent of their need for conviction, which had already been fully satisfied by exploration with dynamic geometry. This independent need for explanation, I believe can be used most effectively as a first introduction to learners of the value of logical reasoning (proof) as shown in some of the examples above.

Proof as a means of discovery

It is often said that theorems are most often first discovered by means of intuition and/or quasi-empirical methods, before they are verified by the production of proofs. However, there are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner. In fact, it is completely unlikely that some results (for example, the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using quasi-empirical methods. Even within the context of such formal deductive processes as axiomatization and defining, proof can frequently lead to new results.

For instance, consider the following example. Suppose we have constructed a dynamic kite with *Sketchpad* or *Cabri* and connected the midpoints of the sides as shown in Figure 2 to form a quadrilateral *EFGH*. Visually, *EFGH* clearly appears to be a rectangle, which can easily be confirmed by measuring the angles. By grabbing any vertex of the kite *ABCD*, we could now drag it to a new position to verify that *EFGH* remains a rectangle. We could also drag vertex *A* downwards until *ABCD* becomes concave to check whether it remains true. Although such continuous variation can easily convince us, it provides no satisfactory explanation why the midpoint quadrilateral of a kite is a rectangle. However, if we produce a deductive proof for this conjecture, we immediately notice that the perpendicularity of the diagonals is the essential characteristic upon which it depends, and that the property of equal adjacent sides is therefore not required. (The proof is left to the reader).

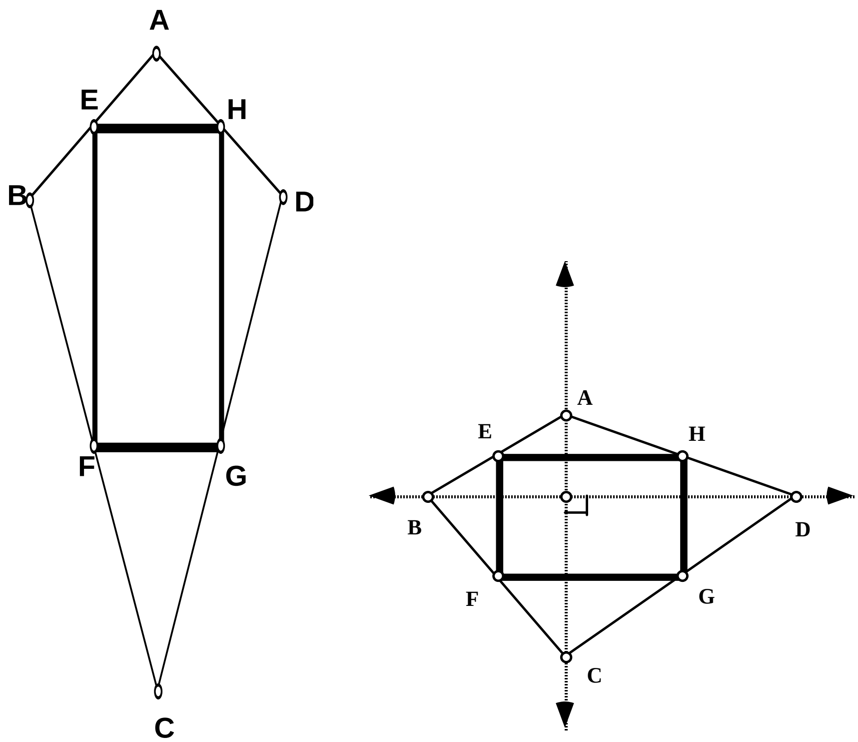

Figure 2

In other words, we can immediately generalize the result to any quadrilateral with perpendicular diagonals (a perpendicular quadrilateral) as shown by the second figure in Figure 2. In contrast, the general result is not at all suggested by the purely empirical verification of the original hypothesis. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since we would probably have restricted our investigation to the familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and isosceles trapezoids.

Proof as a means of systematisation

Proof exposes the underlying logical relationships between statements in ways no amount of quasi-empirical testing or pure intuition can. Proof is therefore an indispensable tool for systematizing various known results into a deductive system of axioms, definitions and theorems. Some of the most important functions of a deductive systematization of known results are discussed extensively in De Villiers (1986).

Although some elements of verification are obviously also present during any systematization, the main objective clearly is not "*to check whether certain statements are really true*", but to organize logically unrelated individual statements that are already known to be true into *a coherent unified whole*. Due to the global perspective provided by such simplification and unification, there is of course also a distinct element of illumination present when proof is used as a means of systematization. In this case, however, the focus falls on global rather than local illumination. Thus, it is in reality misleading to say at school when proving self-evident statements such as that the opposite angles of two intersecting lines are equal, that we are "*making sure*". Mathematicians are actually far less concerned about the truth of such theorems, than with their systematization into a deductive system.

Rather than providing students with ready-made definitions, they should also at some point be engaged in defining some mathematical concepts themselves, i.e. Freudenthal's idea of "*local axiomatization*". Suppose for example we want to formally define the concept of rhombus, then we might proceed by first evaluating the following possibilities by construction and measurement on dynamic geometry (compare Govender & De Villiers, 2002):

(a) A rhombus is any quadrilateral with perpendicular diagonals.
(b) A rhombus is any quadrilateral with perpendicular, bisecting diagonals.
(c)A rhombus is any quadrilateral with two pairs of adjacent sides equal.

Such an empirical investigation easily shows that the first and last ones above are deficient, but no matter how we drag the rhombus constructed according to the specifications in the second case, it always remains a rhombus. This implies that the conditions contained in (b) are sufficient, and that one should be able to accept this statement as a formal definition, and logically derive (prove) all the other properties of a rhombus as theorems (e.g. all sides are equal, etc.)

Conclusion

When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like *Sketchpad* or *Cabri*, they have little need for further conviction or verification. So verification serves as little or no motivation for doing a proof. However, it is relatively easy to solicit further curiosity by asking students *why* they think a particular result is true; that is to challenge them to try and *explain* it. Students usually admit that inductive verification merely confirms; it gives no satisfactory sense of illumination, insight, or understanding into how the conjecture is a consequence of other familiar results. Students therefore find it quite satisfactory to then view a deductive argument as an attempt at explanation, rather than verification.

It is also advisable to introduce students early on to the discovery function of proof and to give attention to the communicative aspects throughout by negotiating and clarifying with one'sstudents the criteria for acceptable evidence, the underlying heuristics and logic of proof. The verification function of proof should be reserved for results where students have experienced some genuine doubts. Lastly, in real mathematics, the purely systematization function of proof comes to the fore only at an advanced stage, and should therefore be withheld till the very end.

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