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Disadvantages of Multiple Choice Tests and Possible Ways of Overcoming Them

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1. Introduction

Teaching large numbers of students, lecturers are often forced to use multiple choice exams. Some lecturers even prefer multiple choice tests because they provide objective grading.
On the other hand multiple choice tests have many disadvantages. It is difficult to check the students' knowledge of the theory, and their understanding of the proofs.
In this paper we would like to share with readers our experience in overcoming such difficulties. In Section 2 we discuss the question how we can check the students understanding of the ingredients of a proof (one that was given in class or even a new proof). In Section 3 we suggest a way to check if the students understand the exact meaning of the conditions of a theorem. We do it by presenting wrong proofs in a multiple choice test format. The paper concludes with a discussion.
These topics were discussed in our previous papers [2,3,5]. The examples in this paper are new. Other related papers are also mentioned in the references.

2. Proofs

It is well known that students need a good theoretical knowledge in Mathematics in order to be able to solve practical problems. It is also common knowledge that proving theorems and formulas gives students a much better understanding of the subject than simply providing them with "cook book" formulas and algorithms.
Examinations are an important part of the learning process. Unfortunately, students do not pay enough attention to the theoretical part of the course if the test contains only computational problems. In many cases the exams are given in multiple choice form so the problem is how to check the students understanding of the theory in multiple choice tests.
We present a partial solution to this problem by using the proofs of Fermat's theorem, the Newton - Leibniz formula, and l'Hopital theorem, as examples.
In the examples the students are given a theorem and its proof, followed by a list of theorems and definitions. In the proof, several steps are outlined and students are asked to point out which theorems or definitions are used in each step.

Example 1

Theorem. If $f(x)$ is differentiable at x_0 and x_0 is a local extremum point of $f(x)$ then $f'(x_0)=0$.

Proof:

Step 1: Let x_0 be the maximum point of $f(x)$. Then $f(x)\le f(x_0)$ or $f(x_0+\Delta x)-f(x_0)\le 0$.

Step 2: $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\le 0$ if $\Delta x>0$ and $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\ge 0$ if $\Delta x<0$.

Step 3: $\lim\limits_{\Delta x\to 0^+}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\le 0$ and $\lim\limits_{\Delta x\to 0^-}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\ge 0$.

Step 4: Since $f(x)$ is differentiable at x_0, $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Step 5: $\lim\limits_{\Delta x\to 0^-}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Step 6: $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 0$.

The students have to point out which of the following theorems / definitions are used in proving the steps that are outlined in the proof. It is possible that a given theorem / definition is not used at all.

1st. The definition of a maximum point.
2nd. The definition of a derivative.
3rd. Theorem: $\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$.
4th. Theorem: If $g(x)$ is differentiable in the interval (a,b) and $g'(x) > 0$ in this interval then $g(x)$ increase in this interval.
5th. The definition of the continuity.
6th. The properties of the limits.
7th. Theorem: If $g(x)$ is differentiable at x_0 then $g(x)$ is continuous at x_0.

Students are expected to fill out the table:

Step	Theorem - Definition						
	A	B	C	D	E	F	G
1	x						
2							
3						x	
4		x					
5			x				
6							

Example 2

Theorem: If $f(x)$ is continuous in the interval $[a,b]$ than $\int_a^b f(x)dx = g(b) - g(a)$ where $g'(x) = f(x)$.

Proof:

Step 1: Let $h(x) = \int_a^x f(t)dt$.

Step 2: $h'(x) = f(x)$.

Step 3: Let also $g'(x) = f(x)$.

Step 4: $h'(x) - g'(x) = 0$.

Step 5: $(h(x) - g(x))' = 0$.

Step 6: $h(x) - g(x) = C = const.$ or $h(x) = g(x) + C$.

Step 7: $h(a) = \int_a^a f(t)dt = 0$.

Step 8: $h(a) = g(a) + C$ such that $C = -g(a)$.

Step 9: $h(b) = \int_a^b f(t)dt = g(b) + C$.

Step 10: $\int_a^b f(t)dt = g(b) - g(a)$.

The theorems / definitions are:

1st. The fundamental theorem of the calculus.
2nd. The theorem on the derivative of a sum.
3rd. Properties of the definite integral.
4th. Theorem: If $g'(x)=0$ on (a,b) then $g(x)$ is constant on (a,b).
5th. Lagrange's theorem.
6th. The definition of a derivative.
Remark: We leave it to the reader to add other irrelevant theorems to the list.
Students are expected to fill out a table similar to that of Example 1.
<u>Example 3</u>
<u>Theorem:</u> Let $f(x), g(x)$ be two functions defined on the interval $[a,b]$ and $x_0 \in (a,b)$. If:

a. $f(x), g(x)$ are differentiable on $[a,b]-\{x_0\}$ and continuous at x_0.
b. $f(x_0)=g(x_0)=0$.
c. $g'(x)\neq 0$ for $x \neq x_0$.
d. It exists the limit $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\lambda \neq \pm\infty$.

Then $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lambda$.
We remark that the theorem is true even for $\lambda=\pm\infty$.
Proof:
Step 1: $\frac{f(x)}{g(x)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}$.
Step 2: $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(c_x)}{g'(c_x)}$ when $x<c_x<x_0$ or $x_0<c_x<x$.
Step 3: If $x\to x_0$ then $c_x \to x_0$.
Step 4: $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(c_x)}{g'(c_x)}=\lambda$.
The theorems / definitions are:
1.. Lagrange's theorem.
2.. Cauchy's theorem.
3rd. The definition of the (finite) limit of a function in a point.
4th. Theorem: If $\lim_{x\to x_0} f(x)=y_0$ and $g(x)$ is continuous at y_0 then $\lim_{x\to x_0} g(f(x))=\lim_{y\to y_0} g(y)$.
5th. Theorem: If the functions $f(x), g(x), h(x)$ are defined in a neighborhood of a point x_0 (not necessarily at x_0 itself) and for every $x \neq x_0$ in this neighborhood the inequality $g(x)\leq f(x)\leq h(x)$ holds and if $\lim_{x\to x_0} g(x)=\lim_{x\to x_0} h(x)=L$ then $\lim_{x\to x_0} f(x)=L$.

Students are expected to fill in a table similar to that of Example 1.
The three precedent examples do not test the students ability to prove a theorem (on his own) but it checks the students' understanding of what exactly is used in each step of a given proof and motivates the students to seriously study the theoretical part of the course. We believe that it makes sure that the student understands all parts of the proofs.

<u>3. "Proofs"</u>

In our teaching experience we have found it useful to present, from time to time, to our students wrong proofs and then explain the mistakes. This can be effective in improving the students understanding of the subject and may prevent them from making similar mistakes.
Here are examples of problems given in multiple choice tests. In the following example, several explanations of a mistake are offered. These statements are correct but only one of them is relevant.

Example 4

This is an incorrect proof of the incorrect statement: $\lim\limits_{x\to 0^+} \sin\frac{1}{x} = 0$.

'Proof': Let $f(x)$ be the function

$$f(x)=\begin{cases} 0, & x\le 0 \\ \sin\frac{1}{x}-\frac{1}{x}\sin\frac{1}{x}, & x>0 \end{cases}$$

and let $F(x)$ be a primitive of $f(x)$ on the real axis. Then

$$F(x)=\begin{cases} C, & x\le 0 \\ x\sin\frac{1}{x}+D, & x>0 \end{cases}$$

Since $F(x)$ is differentiable at $x=0$, it is also continuous at the same point. But $\lim\limits_{x\to 0^-} F(x)=C,\ \lim\limits_{x\to 0^+} F(x)=D$ and it follows that the constants $C=D$.

By the definition

$$F'(0-)=\lim_{x\to 0^-}\frac{F(x)-F(0)}{x}=0$$

$$F'(0+)=\lim_{x\to 0^+}\frac{F(x)-F(0)}{x}=\lim_{x\to 0^+}\sin\frac{1}{x}$$

and $F'(0-)=F'(0+)=F'(0)$ we have finally $\lim\limits_{x\to 0^+}\sin\frac{1}{x}=0$.

Among the following statements there is one that explains why this 'proof' is wrong. Please mark it:

a. The function $x\sin\frac{1}{x}$ is a primitive of $\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ for $x\ne 0$.

b. The limit $\lim\limits_{x\to 0} x\sin\frac{1}{x}=0$.

c. If $g'(x)=0$ on the interval (a,b) then $g(x)=const.$ on (a,b).

d. If $g(x)$ is differentiable at x_0, then $g(x)$ continuous at x_0.

e. The primitive of $f(x)$ does not exist in the neighborhood of 0.

(The correct answer is of, course, (v).)

The structure of the next example is different.

Example 5

This is a wrong proof of the correct statement: $(\sin x)' = \cos x$.

"Proof":

Step 1: By the definition of the derivative

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Step 2: $f'(x)=\lim\limits_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}$.

Step 3: By using a trigonometric formula

$$f'(x)=\lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x+\sin\Delta x\cos x-\sin x}{\Delta x}.$$

Step 4: Since $\lim\limits_{\Delta x\to 0}\cos\Delta x=1$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\sin x\cdot 1+\sin\Delta x\cos x-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin\Delta x\cos x}{\Delta x}.$$

Step 5: By using $\lim_{t \to 0} \frac{\sin t}{t} = 1 \quad f'(x) = \cos x$.

Mark the step, where the mistake was made:

a. Step 1;
b. Step 2;
c. Step 3;
d. Step 4;
e. Step 5;
f. There is not any mistake.

(The correct answer is, of course, (iv)).

4. Discussion

All the examples in this paper are taken from Calculus. There are many similar examples in others mathematics courses.

There are many examples of "proof" that can be used in tests. When we started to use (correct) proofs in multiple choice tests, we were worried that it would be difficult to find enough examples. We now feel that there are many examples that can be used.

Another way of checking the understanding of the theory is using questions where the students have to mark if a given statement is true or false. We feel that questions of the type given in this paper shed more light on the students' comprehension of the subject.

Finally, in constructing a multiple choice test there are many problems that should be taken into account. Examples of ill-formulated multiple choice problems and ways of constructing "good" ones are given in [3].

5. References

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