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Dynamic Representations and Prospective Teachers' Understanding of Functions

Beverly J. Ferrucci, Keene State College, USA & Jack Carter, California State University USA

Abstract: This study utilized class worksheets to investigate the effectiveness of using a series of dynamic computer activities to improve prospective teachers' knowledge of functions. Earlier findings indicated that activities involving the construction of functions were demanding for the prospective teachers. Based on these findings, a subsequent implementation of the activities focused on presenting problems that would enable the prospective teachers to describe and construct functions more effectively. Results indicated that the prospective teachers' understanding of activities dealing with domain and range, parity of functions, and composite functions improved from earlier to later instruction. While the prospective teachers' work on activities dealing with functions of circular measures and descriptions of parameters in dynamic graphs demonstrated comparable improvements, no obvious improvement was noted with work on functions of triangular distances and points on Cartesian graphs. The results of the study confirmed reports that dynamic technology allows prospective teachers to explore problems more deeply and, as a result, to develop a more sophisticated understanding of functions.

Recent research (Bloch, 2003; Sajaka, 2003; Yerushalmy, 2000) has shown that technology can enable students to make better connections between symbolic and graphic representations of functions, describe and construct functions more effectively, and demonstrate a clearer understanding of multiple representations of functions. Gay and Keith (2002) have indicated that prospective teachers continue to have difficulty with a wide range of activities related to the construction of functions and that there is a need to create a learning environment that will help with the teaching of functions. Other studies with prospective teachers have found that mathematical software and computer applications allow them to explore problems more clearly and deeply (Heid, Hollebrands, Iseri, Edwards, and Graham, 2002; Zbiek, 1998).

This study extends this research by exploring the use of a computer-generated dynamic system as an enhancement tool for prospective teachers' understanding of functions. The participants were prospective teachers completing a mathematics content course for prospective elementary school teachers. Topics that were taught earlier in the course included (1) linear, probability, and statistical functions; (2) spreadsheets; (3) statistical plots (histograms, scatterplots, and box plots); and (4) graphing calculators (linear functions, statistical plots and fitting linear equations to data). Consequently, this study intended to use computer-generated dynamic representations of functions to deepen and extend the participants' expertise with functions, especially representations, characteristics, and compositions of functions.

There were two prerequisites for the course in which the participants were enrolled. The first concentrated on the real number systems and its applications, while the second focused on geometry. As a result, this study originally intended to connect the participants' prior work in geometry with dynamic representations of geometric functions. However, as it was later realized, only a few participants had gained experience with dynamic geometry software, while others had not even completed the geometry prerequisite.

In this study and in a previous investigation (Carter and Ferrucci, 2003) participants worked individually at computers in a computer lab on Geometers Sketchpad activities designed to investigate aspects of functions and their properties. Worksheets were prepared for each class that contained introductory, sketch-making, and exploratory sections in which the participants had a total of five 110-minute class sessions to complete the activities. Each introductory section described the activity and was followed by the sketching and exploring sections. These sections provided the participants with detailed constructions followed by questions pertaining to functions and their properties.

The first two activities served as an introduction to the types of questions that dynamic coordinate geometry opens for study. During Activity 1 the participants investigated a circle's circumference as a function of its diameter. In Activity 2 they explored the length of the segment that had endpoints on two sides of a triangle and was parallel to the third side as a function of the segment's distance from the third side. These activities were designed as an exploration of the relationships between actual measurements and graphs. In both these activities the underlying functional relationships were linear.

A dynagraph, a type of graphical representation that was developed to promote the comprehension of functions, was used throughout the next five activities. These graphs of functions are typically displayed on a coordinate system whose x- and y-axes are parallel to each other. An advantage of using dynagraphs is that these representations provide a graphical expression of the input-output view of functions as well as help to develop an understanding from input-output machine models of functions and graphs of functions on a Cartesian coordinate system (Goldenberg, Lewis, and O'Keefe; 1992).

Participants in this study used dynagraphs in each of the class sessions as a means of investigating the domain and range of functions, odd or even functions, and composite functions. Specifically, during class session 2 participants experimented with the selection of independent and dependent variables, predicted the graphs of the resultant functions, and checked their predictions by sketching the graphs of the functions. They also identified the numerical inputs and outputs of dynagraphs.

Activities in class session 3 involved the identification, transformation, and comparison of dynagraphs to Cartesian graphs. The activities also presented opportunities to investigate which properties of functions were more evident when graphs were displayed on coordinates with x- and y-axes parallel rather than on perpendicular axes. As the participants matched dynagraphs with the form $y = f(x)$, they described how changing input and output indicators affected Cartesian points. They also identified the domain and range of functions from dynagraphs and Cartesian graphs, while being given further opportunities to build on their initial work with dynagraphs by studying the notions of domain, range, odd, even, and composition as they apply to functions.

The participants explored symmetries in Cartesian graphs and in dynagraphs to determine a function's parity (whether a function was odd, even, or neither) during the fourth class session. After finding the parity of functions from their graphs, they then used this experience to determine parity without using the graphs. To further investigate composite functions, dynagraphs were utilized to model and find the values of other composed functions.

In the fifth and last class session, participants created functions and their accompanying graphs by using iteration of arithmetic operations on the Cartesian coordinates of points and the loci of these iterated points. These activities were designed to use iteration to demonstrate the connection between symbolism of functions and the visual nature of their graphs.

Each participant's performance on the activity sheets was evaluated by a class reader, calculated as a score, and recorded as a percentage. Group 1 consisted of 18 participants who were enrolled in the course during the first implementation of the activity sheets, while Group 2 consisted of 23 participants during the second implementation.

Results

Activities 1 and 2 were designed to acquaint the participants with dynamic geometry and the use of dynamic graphs to study functions. Interestingly, participants' scores on the first two activities showed considerable disparities. In particular, Group 1 participants' mean scores on the first activity were the highest of all the activities, while the mean scores on the second activity were the lowest of all the activities for participants in both groups. This difference may reflect the fact that only some of the participants had previously been exposed to the software.

The main focus of the study centered on the next five activities (Activities 3-7).

These activities were designed to provide participants with a methodical development of the fundamentals of functions and their operations. Participants' mean scores on Activities 3 and 4 closely coincided with the participants' overall class means for the entire course. The mean scores on Activities 5, 6, and 7 demonstrated a gradually rising from the means of the two previous activities. These activities could be regarded as the most intellectually rigorous of the activities as they formed the core of the explorations into the study of functions and their compositions. Activity 8 was intended to provide an alternative dynamic representation of functions as a conclusion to the sequence of activities. The mean scores on Activity 8 were comparable to the participants' overall mean scores for the entire course.

Table 1 shows the overall activity means and class means for the participants in the study.

GROUP 1	OVERALL ACTIVITY MEAN	OVERALL COURSE MEAN		GROUP 2	OVERALL ACTIVITY MEAN	OVERALL COURSE MEAN
Participant1	63.63	69.27		Participant1	66.1	79.7
Participant2	87.8	73.54		Participant2	93.4	87.2
Participant3	63.03	76.03		Participant3	88.7	86.0
Participant4	70.36	75.71		Participant4	92.2	94.4
Participant5	70.81	76.0		Participant5	71.6	58.1
Participant6	79.42	83.07		Participant6	86.5	87.3
Participant7	79.63	88.99		Participant7	50.4	68.1
Participant8	91.73	95.81		Participant8	64.9	87.0
Participant9	46.25	62.09		Participant9	79.4	81.0
Participant10	50.92	76.3		Participant10	84.3	79.0
Participant11	65.98	71.69		Participant11	86.3	91.6
Participant12	66.88	77.29		Participant12	85.9	78.5
Participant13	92.06	83.9		Participant13	75.2	76.7
Participant14	97.64	92.12		Participant14	93.8	88.0
Participant15	40.45	79.15		Participant15	88.9	83.7
Participant16	94.01	92.59		Participant16	86.8	93.1
Participant17	42.43	77.44		Participant17	64.5	78.8
Participant18	55.24	81.94		Participant18	78.7	69.3
				Participant19	25.3	71.1
				Participant20	87.7	81.8
				Participant21	81.6	83.4
				Participant22	78.6	94.0
				Participant23	51.5	78.7
MEAN	69.98	79.61		MEAN	77.2	81.59

TABLE 1: Overall Activity Means and Course Means

In summary, the results indicated that the prospective teachers' understanding of activities dealing with domain and range, parity of functions, and composite functions improved from earlier to later instruction. While the prospective teachers' work on activities dealing with functions of circular measures and descriptions of parameters in dynamic graphs demonstrated comparable improvements, no obvious improvement was noted with work on functions of triangular distances and points on Cartesian graphs.

Further analysis of the individual activity mean scores for the Group 1 participants showed that 50% (9 out of 18) of the scores were lower than the overall activity mean score for Group 1. This same comparison indicated that 35% (8 out of 23) of the scores in Group 2 were lower than the overall activity mean score for that same group. When comparing the overall course means, 61%

(11 out of 18) of the Group 1 participants had scores that were lower than the overall course mean for its group, while 48% (11 out of 23) of the Group 2 participants had lower scores than their group's overall course mean.

Discussion

As previously mentioned, participants in this study should have completed a prerequisite course in geometry that included an introduction to the dynamic geometry software. The fact that a few participants had not completed this prerequisite may have had a considerable effect on their use of the dynamic coordinate geometry as well as an understanding of the related class activities.
In addition, the participants should have completed a course on the real number system and its operations, properties, and applications. One part of this course deals with relations and functions, with specific material on domain, range, and composite functions. However, as was the case with a few participants regarding the geometry course, not all of them may have completed this course prior to, or even concurrent with, the course in which the present study was conducted. Furthermore, as various instructors stress different topics within their courses, some participants who had completed the real number system prerequisite may not have studied in-depth the topics of domain, range, and function composition. This fact may have also affected the participants' performances in this study.
There is little doubt that the proliferation of the use of technological tools in classrooms has impacted mathematics curricula and instructional practices. However, the question of the effectiveness of such technological tools remains debatable. More research is needed to provide information critical in determining how such technological tools could be best used to positively impact student learning, including longitudinal studies that can better assess long-term benefits or disadvantages of their use.
Particularly, the current study points to the need for further research to assess the effectiveness of using the activities presented in this paper for enhancing prospective teachers' understanding of functions and their properties. Just as prospective teachers need experience working with dynamic geometry environments, mathematics teacher educators need to continue to explore ways to teach with dynamic geometry software. These dynamic tools will undoubtedly continue to play significant roles in the reformation of mathematics teaching and learning at all levels.

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