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AN EXPONENTIAL FUNCTION: IS ITS DESCRIPTION NOT PROBLEMATIC?

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Abstract: *The article identifies some questions that could facilitate research in studying the impact of describing* $y = a^x$ *as an exponential function. The impact to be studied would be in relation to the performance of students in applying logarithmic laws, especially against the background of the statement that a logarithmic function is the inverse of an exponential function. The author argues that for the statement to be true, the description of the exponential function should be revisited. Among the reasons for the need to revisit the description is the possible impact the description can have on the students' application of logarithmic laws, as well as the understanding of functions in general.*

1. Introduction

According to many South African textbooks, an exponential function is a function defined by an equation of the form $y = a^x$, $a > 0$; $a \neq 1$. Laridon et. al. (1996) further mention that the function is called exponential because the independent variable x is an exponent.
The immediate question is whether, for consistency sake, the equation $y = x^2$ defines a base function. If the answer is affirmative, shouldn't the functions $y = x^2$, $y = x^3$, or in general the functions of the form $y = x^b$, b a constant, which are normally called power functions, be called base functions? If a base function is a power function, does this not cloud the meanings of base and power for the students?
We discuss the description of an exponential function under the following topics:

1. Theoretical basis for worrying about this topic.
2. Possible impact (to the learners) of sticking to the description of $y = a^x$ as an exponential function.
3. Description of an exponential function that would eliminate the possible impact in 2 above.
4. Teaching approach to identifying the inverse of a logarithm.

1. Theoretical basis for worrying about the topic of discussion.

The problem with the exponential function as it is described is that this description is not consistent with the manner of describing other 2-dimensional functions. One would expect an exponential function to be characterized by exponents being represented along the y-axis. However, the exponential function graph has exponents represented along the x-axis. This means that in order for a student to fully understand the exponential function as it is described, the student would have to abandon his/her conventional understanding of representation of 2-D functions.
There is literature survey to suggest the need for consistency when it comes to dealing with mathematical principles and/or concepts. Hiebert and Carpenter (1992) define mathematics understanding in terms of what they call external and internal mental mathematical representations. They claim that a mathematical idea or a procedure or fact is understood if it is part of internal mental network.
Past experiences, according to Hiebert and Carpenter (1992) create mental networks that the learner uses to interpret and understand new experiences and information. Davis (1984) asserts that the process of recognising something is presumably the process of matching up input information with an appropriate previously created representation structure. Glaser (1984:120) mentions that people continually try to understand and think about the new in terms of what they already know. According to Ausubel (1968:vi), the most single factor influencing learning is what the learner already knows. Understanding, according to Hiebert

and Carpenter (1992), increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring.
What the above literature tells us is basically the need for consistency as a building block for understanding. We argue that the naming of an exponential function goes against this notion of understanding. Consequently we ask whether there will not be unintended results if we expect the students to understand this description of exponential function.

2. Impact of describing $y = a^x$ *as* the exponential function.

The following are some of the misconceptions the students make in the application of logarithms:

$$1.(\log_3 x)^4 = 4\log_3 x$$
$$2.\ln x^2.\ln x = \ln(x^2 \times x)$$
$$3.\log(1+\frac{2}{y}) = \log 1 + \log\frac{2}{y}$$
$$4.\sqrt{\ln y} = \ln y^{\frac{1}{2}}, x\ln y \times (\ln y)^{\frac{1}{2}} = \frac{1}{2}x(\ln y)^2$$
$$5.\ln(3^{2x-1} \times 4^{x-1}) = \ln 3^{2x-1} \times \ln 4^{x+1}$$
$$6.\ln 5x^{-2} = -2\ln 5x$$
$$7.\ln y^{\frac{1}{2}} = -\ln y^2$$

The questions we are asking are, among others:

- Do the students know that *a logarithm is an exponent*?

We probably all agree that a logarithm of a number to a base is an *exponent* to which the base must be raised to equal the number:

$$\boxed{a^b = c \Leftrightarrow \log_a c = b.}$$

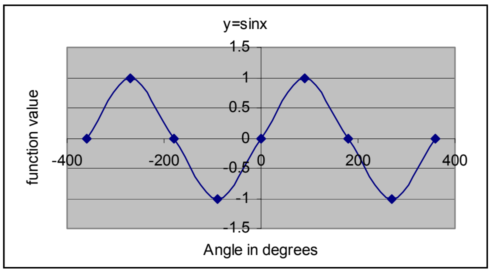
The question here is that if the students pictured some exponent α each time they see $\log_3 x$ (for each x), would it not occur to them that $(\log_3 x)^4$ means α^4 and that $4\log_3 x$ means 4α, so that they can see the inequality of the two?

- If they *do* know that *a logarithm is an exponent*, why would they not be expected to think (from the definition of a logarithm as an exponent, not from their understanding of what *is meant* by exponential function) that a logarithmic function is an exponential function?
- Would the definition of an exponential function as a function defined by the equation $y = a^x$ not be received by the students *within their understanding* of an exponential function as a function of exponents?

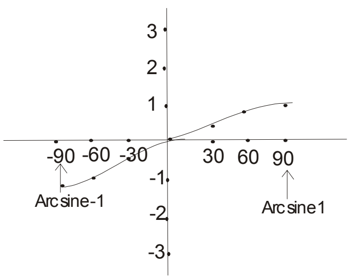
In many of the textbooks that introduce $y = a^x$ as an exponential function, no effort is made to address the possible interpretation of an exponential function as a function of exponents. The basis of the argument here is that the statement that a logarithmic function (i.e. a function of logarithms) *is the inverse of an exponential function* (without explicitly mentioning that by exponential function you do *not* mean a function of exponents) may appear to the students to suggest that exponents and logarithms are separate concepts.
Does the separation not cloud the student's ability to see a logarithm as an exponent, and hence a logarithmic function as an exponential function? Can we categorically exclude the

clouding as a possible contributory factor in the poor showing of the students in the application of logarithmic laws? This is a research topic.
One may argue that the students should know that an exponential function is not a function of exponents. What would be the justification for the argument, if the same students have been taught that, for instance, a *sine function,* is a function of *sine values,* a *quadratic function (a parabola)* is a function of *values of quadratic expressions,* that a *power function* $y = x^b$ is a function of *values of powers* x^b?
In other words, what would be the justification of describing an exponential function in a manner that is inconsistent with the manner in which other functions are named? Would it not have the potential to distort the students' understanding of functions? The following is an example to concretize our conventional manner of describing a function.

Function value in graphical representation in a Cartesian Plane.
In a two-dimensional graphical representation, using the Cartesian plane, we normally use the y-value to indicate the value of a function (if the graph itself represents a function). We give some familiar examples in order to highlight the fact that we usually name functions according to their values.
2.1. The sine function $y = \sin x$, $-360° \le x \le 360°$



We call this function the sine function because the values along the y-axis are the sine values of the angles.
For the x-values $-90° \le x \le 90°$, the x-axis actually represents arcsiny, as the following diagram shows.

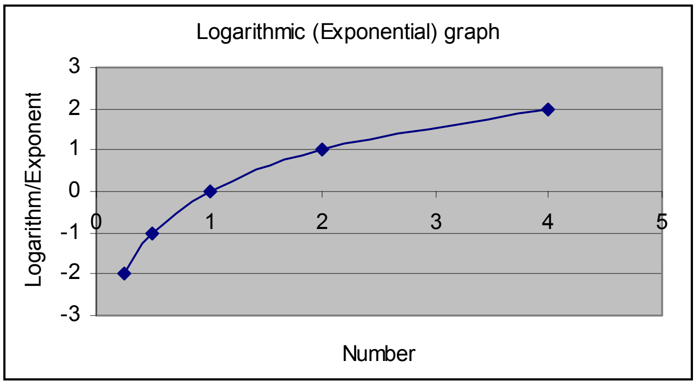


In other words, it would make sense for us to call the x-axis, in this particular case (where $-90° \le x \le 90°$), the arcsine axis. The x- and y-axes represent different quantities, with the function value usually represented on the y-axis.

3. If an exponential function were to be considered to be a function of exponents.
We now look at the implication of describing an exponential function as a function of exponents. The following diagram shows that an exponential function would then be what the textbooks describe as the logarithmic function.

3.1. The logarithmic function $y = \log_a x, a \succ 1$.

The y-value is an exponent, meaning that a logarithmic function is actually a function of exponents. We look at a specific example $y = \log_2 x$. For each chosen x value, the corresponding y-value is an exponent as the following graph shows.



The above graph shows that a logarithmic graph *is* the graph of exponents. Calling a logarithmic function an exponential function is consistent with:

- the definition of a logarithm as the exponent, as well as
- the notion of a function value,

but generally *inconsistent* with the notion of a logarithmic function as being an inverse of an exponential function, unless in a reflective sense of equality.

The research question one would ask is whether calling an exponential function (not the one described in the textbooks) a function of exponents would not have the following advantages:

3.1.1. Reinforcing the fact that an exponent is a logarithm and hence facilitating the correct application of logarithmic laws.

3.1.2. Reinforcing the conventional manner of naming functions.

This notion of describing an exponential function as a function of exponents would have the following repercussions:

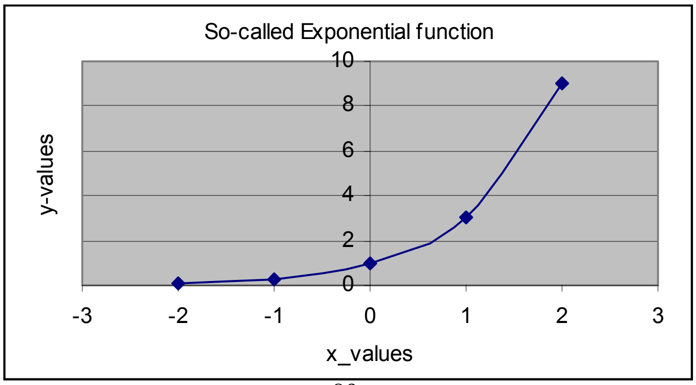
3.1.3. The graph purported to describe the exponential function would have to be renamed.

3.1.4. The statement that a logarithmic function is an inverse of an exponential function would also need to be revisited, in the sense of replacing the term 'exponential'.

The question that we would now focus on is: What would we have to call the so-called (text-book) exponential function? We now focus on that.

3.2. The so-called *exponential* function $y = a^x$.

The following is an example of such a function: $y = 3^x$.



In order for the above function to comply with notion of a function value as represented along the y-axis, it needs to be called a *power* function, in the sense of a power being defined as a base raised to an exponent. We normally talk of adding exponents when multiplying *powers* of the same base. It is within the same context that the term power is used here.
The following are some of the reasons why the so-called exponential function should be called a power function:

- The properties of what the textbooks call an exponential function are those of a power function:
  - Logarithmic tables are sometimes used, where an *anti*logarithm of an exponent is the power, or its numerical equivalent. An antilogarithm is the inverse of a logarithm, the same way that an antiderivative (i.e. an integral) is an inverse of a derivative. By mentioning that the so-called exponential function is the inverse of the logarithmic function, it appears what is being described as an exponential function is actually a power function.
  - The scientific calculators have most of the functional *inverses* accessed by keying in the second function button. The following are examples:

Function keyed in	**The inverse accessed through 2nd Function**
$\cos x$	$\cos^{-1} y$
$\tan x$	$\tan^{-1} y$
$\log x$	10^x
$\ln x$	e^x

  - While it may be tempting to refer to $a^x = b$ as an exponential equation, it would be difficult to justify reference to, for instance, 10^x, as an exponential term or expression. It is clearly a power as we usually define the power. Note that in drawing the function $y = 10^x$, we actually plot the exponent x against *the power* 10^x. So the function value is 10^x, which is a power. This is further supported by the scientific calculator, where 10^x is reflected as the inverse of a logarithm. This shows that an inverse of a logarithmic function is a power function, and not the stated exponential function.
  - We have already talked of $y = x^b$ as a power function. How would this impact on also naming a^b a power function? Would this not lead to another anomaly? We do not think so because, in the same way that power functions $y = x^3$ and $y = x^4$ would not give the same functions and yet they are both classified as power functions, there is no obvious contradiction in talking about two *different categories* of power functions – the one with the *base being a variable*, and the other having a *constant base*. What is significant here is the naming of both as power functions is not done at the expense of standing mathematical conventions, and consequently may not have unintended misconceptions.
- So if the textbooks were to call what they have been referring to as an exponential function a power function, then they would be logically correct to say a power function is an inverse of a logarithmic function.

There may be some concerns that can arise out of having to abandon the idea that logarithmic and exponential functions are inverses of each other. We highlight some of them.

3.3. Exponential laws vs logarithmic laws.

- **Is the usual teaching of exponential and logarithmic laws as separate topics justified?**

- In the case where we usually discuss what we have been calling *exponential laws*, our focus has always been on looking at the behaviour of exponents when multiplying and/or dividing *powers* of the same base, or raising powers to certain exponents. The so-called exponential laws are actually laws of operations (multiplication, division, raising to exponents) on *powers*, or power laws. However, since power operation result into a certain pattern of behaviour of exponents, we can retain the term *exponential laws.* It is however important that exponential laws should not be construed to imply *laws of exponential equations,* but rather as exponential laws of *power operations.*

The exponents and logarithms can be handled separately, as long as their common meaning can emerge unambiguously. For instance, if one considers the exponential law $a^x \times a^y = a^{x+y}$, then one could immediately verbalise this to mean that the sum $x + y$ of exponents x and y is the exponent of the product of powers a^x (=M, say) and a^y (=N,say). In other words, the logarithmic interpretation of the verbalization would be $\log_a M + \log_x N =$ $\log_a MN$. It may initially be difficult for the students to change from exponential to logarithmic form, but once they have succeeded, their conceptualization, and subsequent application of, logarithms, could possibly be enhanced.

4. Teaching approaches: Inverse relations

The reason for including the topic on approach to teaching inverse relation is to further highlight that naming the so-called exponential function *power function* would make it easier to use the input-output model to introduce the power as the inverse of a logarithm. The use of the model would then be *consistent* with *handling* any other inverse relation.

The input-output model is the method in which learners are facilitated to discover the relationships, and is characterized by active involvement of the learners in the lesson. The following are some of the examples of the use of this approach.

4.1. Cosine and inverse cosine functions.

Here learners are required to indicate out-put and inputs as per following diagram.

Input	Cosine of input	Output
i). 0°		i). 1
ii) .		ii) $\frac{1}{2}$
iii) 45°		iii)
iv)		iv) $\frac{\sqrt{3}}{2}$

The learners are aware that in one case they identify output, given input, and in the other, input, given the output.

Upon request for them to indicate how they will write the "backward" movement from output to input, they will come up with the response that can, through facilitation, be refined to

Inverse cosine of the output.

Notations like $\arccos\frac{\sqrt{3}}{2}=\frac{\pi}{3}$, or $Cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$ can then follow. The method will also help in dispelling some of the misconceptions like $\cos^{-1}x=\frac{1}{\cos x}$, in the sense of the learners being conscious that in determining $\cos^{-1}x$, they are actually looking for an angle.
The similar type of argument can be extended to the introduction of integration as the inverse of differentiation. The following is an example of how learners can be led to the discovery of the formula

$$\int f'(x)[f(x)]^n\,dx=\frac{[f(x)]^{n+1}}{n+1}+c$$

4.2. Differentiation and integration: $\int f'(x)[f(x)]^n\,dx=\frac{[f(x)]^{n+1}}{n+1}+c$

Input	Differentiate	Output
i). $\frac{(x^2+1)^2}{2}$		i).
ii) .		ii) $3x^2(x^3-2)$
iii) $\frac{(x^2-x+1)^4}{4}$		iii)
iv)		iv) $\sin\theta\cos^3\theta$

This may require a number of input – output problems for the learners to conjecture the formula. The output-input movement could then be introduced as
antiderivative, inverse differentiation, or integration.

4.3. Logarithm and its inverse.

Input	Logarithm of input	Output
i). 10		i). 1
ii) .		ii) 3
iii) 10 000		iii)
iv)		iv) 2

If the learners were to be exposed to this exercise, using a calculator without being told what a logarithm is, they should be able, through facilitation, to observe that

- the input column consists of *powers* of ten, while
- the output column consists of exponents, despite the fact that they had to press log button to access this exponent.

This would be sufficient to enable the learners to realize that:
the inverse of a logarithm is the power, and not the exponent.

5. Conclusions

Unless an exponential function as it is usually drawn, is properly redefined to be a power function, the statement that an exponential function is an inverse of a logarithmic function may possibly clouds the understanding of the concept of logarithms, and possibly dispose learners to some misconceptions.

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