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GAMES AND PROBABILITIES

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Abstract. We propose an introduction in *Combinatorial Probabilities*, using some problems based on a number of games which are more or less known, in the world of *Mathematics*.
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PRELIMINARIES

Probability Theory is generally introduced with difficulty in the lectures for the university students and only in the superior study-years, because it requires the knowledge of many other branches of *Mathematics* and especially the theories of *Integrability* and *Measurability*. This is the right way for the faculties of *Mathematics* and there is no impediment in the appearance of *Probability Theory* lecture towards the final study years. On the other part, in all technical faculties, the study of *Probabilities* is required faster, by other lectures as *Physics*, *Chemistry* or *Technical disciplines*. Therefore, the direct way in the preseting of *Probability Theory* is in link with that which is considered *classical* or *Combinatorial*, for the discrete case, respectively, in link with that which uses *Stieltjes integral*, for the continuous case. Thinking of the fact that our daily life imposes mostly discrete phenomenons, which are found in technical, economical or social events, I maintain the idea that the *Probabilities* could be introduced since the beginning years of the university studies, and this, after the moment when the *counting methods* will be well controlled. Therefore, a lecture as this, could be named *Combinatorial Probabilities*, as a reference book of the great mathematicians P. Erdös and A. Rényi.

BACKGROUND TOOLS

In the following, we shall use the classical notions of *Combinatorics* (see [9]) and also, those come from *elementary Probabilities* (see [5], [8]). For a *random event* A, its probability, is:

$$\mathrm{P}(\mathrm{A}) = \frac{success\ numbers}{possibility\ numbers} \quad (1),$$

and the probability, of the *complementary event* $\overline{A}$, is:

$$P(\overline{A}) = 1 - P(A) \quad (2).$$

The probability to realize the random event A, if the random event B realizes, is given by the *conditional event* A/B, being:

$$P(A/B) = \frac{P(A \cap B)}{P(B)},\ P(B) > 0 \quad (3).$$

We must also remember, that through *hypergeometric scheme* (see [3]), we could compute the probability $P(\alpha, \beta)$ of the drawing without return, for α white balls and β black balls, when in an urn, there initially are *a* white balls and *b* black balls:

$$P(\alpha, \beta) = \frac{\binom{a}{\alpha}\binom{b}{\beta}}{\binom{a+b}{\alpha+\beta}},\ a \geq \alpha, b \geq \beta \quad (4).$$

In the next rows, we shall use the *incompatible events* notion, for 2 random events A and B, which there are in the situation $A \cap B = \phi$, from where:

$$P(A \cup B) = P(A) + P(B) \quad (5).$$

If the same events A and B are *compatible*, namely $A \cap B \neq \phi$, it occurs the relation:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (6).$$

In the case of *n random compatible* events, we shall use *Poincaré relation*, as a consequence of the *Combinatorial Inclusion – Exclusion Principle*, and so:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P\left(A_i \cap A_j\right) + \sum_{1\le i<j<k\le n} P\left(A_i \cap A_j \cap A_k\right) + \cdots +$$
$$+ \cdots + (-1)^{n-1} P\left(A_1 \cap \ldots \cap A_n\right) \qquad (7).$$

We shall go on remembering the *random discrete variable* notion. Therefor, let consider a finite or numerable set Ω, the adjacent *finite* or *borelian probability field* $(\Omega, \aleph, \Pi)$ (see [8]), where $\aleph \subseteq \Pi(\Omega)$, and $P : \aleph \to [0,1]$ is a *probability function*. Now, we define the *random discrete variable*, as a function $X : \Omega \to \mathbf{R}$, where:

$$X(\Omega) = (x_i)_{i\in I},\ x_i \ne x_j \text{ for } i \ne j \text{ when } I \subseteq \mathbf{N}.$$

If the events A_i are given by:

$$A_i = \{\omega \in \Omega \mid X(\omega) = x_i\},\ (\forall) i \in I,$$

then, their probabilities are:

$$p_i = P(A_i) = P(\{\omega \in \Omega \mid X(\omega) = x_i\}), (\forall) i \in I \qquad (8).$$

So, it follows that the *discrete random variable* will be denoted by:

$$X : \begin{pmatrix} x_i \\ p_i \end{pmatrix},\ p_i \ge 0, (\forall) i \in I, \text{ when } \sum_{i\in I} p_i = 1 \qquad (9).$$

Its *expectation* $E(X)$ is defined as:

$$E(X) = \sum_{i\in I} x_i p_i \qquad (10).$$

THE MAIN ARGUMENTS, BY PROBLEMS

The beginnings of *Probability Theory* are in connexion with the games and date since more then 4 centuries and half. So, at the frontier between the first and the second part of the XVI-th century, Cardano wrote „*The book about playing dice*" („*Liber de Ludo Aleae"*), although its publishing was realized a century later, around the year 1663. Actually, this is the period in which the problems, risen by games, are presented to be solved by some of the outstanding mathematicians of that period, and who are considered precursors for *Probability Theory*. Between all those who analysed games through *Mathematics*, the queen science, we could quote personalities as: Pascal, Fermat, Huygens, Bernoulli, Laplace. More, Laplace was the first who noticed „It is remarquable, as a branch of the science which started, analysing games, became the most important method of the human knowledge."

In the following, we shall try to prove that using various games (dice, playing cards, Loto etc.), which have a *discrete domain*, then, *Combinatorial Probabilities* could be introduced in the most normal manner. Surely, we shall begin our step, presenting the first *Probability Theory* problem, given to Pascal in 1654, by the French knight, De Méré.

Problem 1. (*De Méré & Pascal*). A player and a bank wager on equable stakes. Study the gain possibilities, when the player wants to obtain:

i) at least a time, a 6, throwing a die, 4 times;

ii) at least a 6 double, throwing 2 dice, 24 times.

Solution. i) We shall consider the event:

A_1 = {the non appearance of the face 6, in 4 throwings}.

With this, we must find $P(\overline{A_1})$. Let firstly notice that the number of possible cases is given by the cardinal of the set:

$$M_1 = \{f \mid f : \{t_1, t_2, t_3, t_4\} \to \{1,2,3,4,5,6\}\},$$

where *f* are defined on the set of all 4 throwings and with values on the die faces. Obviously, $|M_1| = 6^4$.

For the number of success cases, eliminating the face 6, we shall compute the cardinal of the set:

$$M_2 = \{g | g : \{t_1, t_2, t_3, t_4\} \to \{1,2,3,4,5\}\},$$

which surely is $|M_2| = 5^4$. Therefore:

$$P(\overline{A_1}) = 1 - P(A_1) = 1 - \left(\frac{5}{6}\right)^4 \cong 0{,}518.$$

ii) Let the event:

$$A_2 = \{\text{the non appearance of a 6 double, in 24 throwings}\}$$

be. We must find $P(\overline{A_2})$. For this, when 2 dice are thrown, let note that their value set is given by:

$$\{(1,1), \ldots, (1,6), \ldots, (6,6)\} = \{1,2,\ldots,36\}.$$

Hence, the number of the possible cases appears from the cardinal of the set:

$$M_3 = \{u \mid u : \{t_1, t_2, \ldots, t_{24}\} \to \{1,2,\ldots,36\}\},$$

being $|M_3| = 36^{24}$.

For the success cases, eliminating the 6 double, we shall compute the cardinal of the set:

$$M_4 = \{v \mid v : \{t_1, t_2, \ldots, t_{24}\} \to \{1,2,\ldots,35\}\},$$

namely $|M_4| = 35^{24}$. Then, it occurs:

$$P(\overline{A_2}) = 1 - P(A_2) = 1 - \left(\frac{35}{36}\right)^{24} \cong 0{,}492. \blacksquare$$

Remark 1. The situation *i)* of *Problem 1* is favourable to the player, while *ii)*is favourable to the bank. ■

Problem 2. (*De Méré & Pascal*). Two persons take part at a game with more stages quoted one point, everyone. The first who reaches 3 points is the winner. Some unforeseeable reasons impose that the game be stopped at a score 2-1. Study the share of the initial stake between the players.

Solution. The stake will be proportional shared with the gain of every player, when the game would be continuated. For this, let notice that the winner would be decided in at most 2 stages. Let also suppose that the first player, *I*, is in advantage with the score 2-1. In the hypothetical 2 stages, there are the next 4 equal possible situations:

$$(I, I), (I, II), (II, I)(II, II).$$

In each of them, the first, respectively the second position shows the winner in the first, respectively the second stage. Therefore, *I* has 3 favourable situations and *II* has only one to be declared the winner. Hence, the share probabilities of the stake, are:

$$P(I) = 3/4 = 0{,}75;\ P(II) = 1/4 = 0{,}25\ . \blacksquare$$

Remark 2. It is very important to notice that the stake is not proportional shared with the score, but with the effective gain probability, come from the possibility to continuate the game until its normal finish. ■

Problem 3. (*L. Modan*, see [2]). A pack contains 53 playing cards, from which 13 are jokers. The pack is used in a game with 6 players and in which a joker is at seeing, on the table. Every player receives a card. Study the probability that at least a player could have a joker.

Solution. The 6 players could have in their cards 1, or 2,…, or 6 jokers. Using the *hypergeometric scheme*, namely (4), and keeping account that 6 cards, from 52, will be chosen in $\binom{52}{6}$ ways, it occurs the next probability:

$$P = \sum_{k=1}^{6} \frac{\binom{12}{k}\binom{40}{6-k}}{\binom{52}{6}} \cong 0{,}791. \ \blacksquare$$

Problem 4. (*L. Modan*, see [1]). At a game are drawn out with return, 7 balls inscriptioned from 0 to 9, in view of the obtaining an initial 7 digit number. The players use 7 digit numbers, randomly chosen. They are in gain if they shall obtain all the initial number, or its last 6 digits or, …, or so on, its last 3 digits. Study the gain probability of a player who uses only a number.

Solution. For every $i \in \{3,\dots,7\}$, let consider the next events:

A_i = {the appearance of the last *i* digits from the initial drawing number}.

We notice that $A_i, (\forall) i \in \{3,\dots,7\}$ are incompatible, because the appearance of *i* digits imposes a gain which is different from that of the situation with *i-1* good digits. So, with (5), the gain probability of a player is:

$$P(A_3 \cup \ldots \cup A_7) = \sum_{k=3}^{7} P(A_k) \qquad (11).$$

Farther on, let remark that from 10 digits, all the numbers, with *i* digits, when $i \in \{3,\dots,7\}$, are exactly as the cardinal of the following set:

$$M = \{ f \mid f : \{1,\dots,i\} \to \{0,\dots,9\} \}.$$

Because $|M| = 10^i$, we have:

$$P(A_i) = 1/10^i, (\forall) i \in \{3,\dots,7\},$$

and so on, with (11), it occurs:

$$P(A_3 \cup \ldots \cup A_7) = \frac{1}{10^3} + \cdots + \frac{1}{10^7} = \frac{1}{10^3} \cdot \frac{1 - 1/10^5}{1 - 1/10} = 0{,}0011. \ \blacksquare$$

Remark 3. The anterior *Problem 4* is the *mathematic model* of the game *NOROC* (*LUCK*) of the *Romanian Autonomous Corporation Lottery*. ■

Problem 5. (partial enunciation of *Problem III*, pg. 22, from [6]). Un urn U contains 4 black balls (b) and 2 white balls (w). The urn belongs to a bank which proposes the following game to the interested persons. Simultaneously, the player draws out, without return, 2 balls, from U. The next situations are possible:

- when the balls are black, the player gains $\alpha > 0$ euros and the game finishes;
- when the balls are white, the player loses 6α euros and the game finishes;
- when the balls have different colours, the player draws out, without return, another 2 balls, and:
  - if the both balls are black, he gains $\beta > 0$ euros, after this, the game will be finished;
  - if the balls are not black, he loses 3 euros, after this, the game will be finished.

Let G be the random variable which sketches the gains of the player.

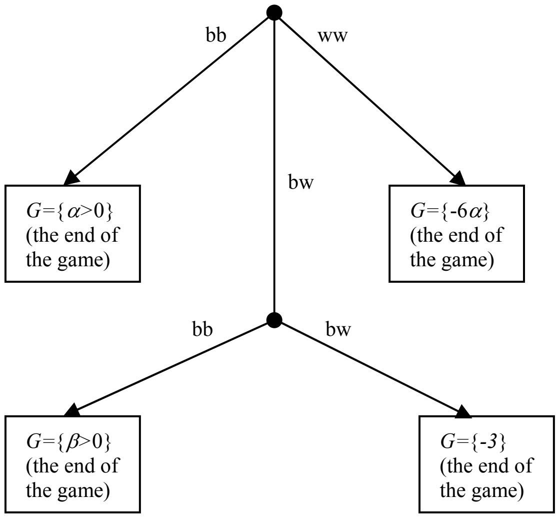
i) Describe the tree corresponding to all possible gains.

ii) Compute the probability *q* when 2 black balls are drawn out the second time, if 2 different colour balls was obtained in the first drawing.

iii) Compute the probability $P(\{G = \beta\})$ for the gain of β euros.

iv) Find the probability law of the random variable *G* and compute β, so that the game would be equitable.

Solution. i) The tree corresponding to the random variable *G* of the gains, is the following:



ii) After the first drawing, without return, in the urn there are 4 balls, from which 3 black and one white. At the second drawing, 2 black balls could be chosen in $\binom{3}{2}$ manners, while 2 different balls, from 4, could be chosen in $\binom{4}{2}$ manners. Hence, with (1), we have:

$$q = \frac{\binom{3}{2}}{\binom{4}{2}} = \frac{3}{6} = \frac{1}{2}.$$

iii) Let A and B be the following events:

A = {the obtaining of 2 different balls at the first drawing},
B = {the obtaining of 2 black balls at the second drawing}.

Then, the gain probability for β euros, with (3), is:

$$P(\{G=\beta\}) = P(A \cap B) = P(B/A) \cdot P(A) = q \cdot P(A) \qquad (12).$$

Surely, with (1), we have:

$$P(A) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} = \frac{4 \cdot 2}{15} = \frac{8}{15} \qquad (13).$$

Now, with (12) and (13), it holds:

$$P(\{G=\beta\}) = \frac{1}{2} \cdot \frac{8}{15} = \frac{4}{15}.$$

iv) With (1), we also have:

$$P(\{G=\alpha\}) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{2}{5},\ P(\{G=-6\alpha\}) = \frac{\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15},$$

$$P(\{G=-3\}) = 1 - P(\{G=\alpha\}) - P(\{G=\beta\}) - P(\{G=-6\alpha\}) = 1 - (2/15 + 8/15 + 1/15) = 4/15.$$

Therefore, the *probability law*, or the *distribution* of G, is the next:

$$G : \begin{pmatrix} -6\alpha & -3 & \alpha & \beta \\ 1/15 & 4/15 & 6/15 & 4/15 \end{pmatrix}.$$

We remember here, that a game is *equitable* when the *expectation* E, of its associate random variable, is 0. So, in our case, we must imposes as $E(G) = 0$. Using (10), it occurs:

$$E(G) = \frac{1}{15}(-6\alpha - 12 + 6\alpha + 4\beta) = \frac{4(\beta - 3)}{15}.$$

Now, we could decide that the *equitability condition* is obtained for $\beta = 3$. ■

Problem 6. (*L. Modan*, see [7]). Let M be a numerable set of initial friend couples which participate at a dance competition. The girls and the boys are separated in 2 rooms. Every time, in front of the jury, a girl and a boy, randomly chosen, dance in couple, and surely, they could not be known persons. Study the probability as, at least, one couple dancing would be represented by an initial friend couple and prove that this probability value is less than 1/2.

Solution. For every $i \in \mathbf{N}^*$, let consider the following events:

A_i = {the initial couple i dances in front of the jury}.

In this way, the asked probability, by the problem, will be:

$$P\left(\bigcup_{i\in\mathbf{N}^*} A_i\right) \qquad (14).$$

Firstly, we shall evaluate $\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right)$ using *Poincaré formula*, given by (7). Therefor, with (1), let notice that:

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n},\ (\forall) i \in \mathbf{N}^*,$$

$$P(A_i \cap A_j) = \frac{(n-1)!}{n!} = \frac{1}{(n-1)n},\ (\forall) i \neq j, \text{ from } \mathbf{N}^*,$$

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$$P(A_1 \cap \ldots \cap A_n) = \frac{1}{1 \cdot 2 \cdot \cdots \cdots \cdot n} = \frac{1}{n!}.$$

These anterior relations permit us to compute:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \frac{1}{n} - \sum_{1\le i<j\le n} \frac{1}{n(n-1)} + \cdots + (-1)^{n-1}\frac{1}{n!} =$$

$$= \binom{n}{1}\cdot\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \cdots + (-1)^{n-1}\frac{1}{n!} =$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{n!} \qquad (15).$$

On the limit, in (15), and using from [4], the series:

$$\sum_{n\geq 1}(-1)^{n-1}\frac{1}{n!}=\frac{1}{e},$$

for (14), we shall find:

$$P\left(\bigcup_{i\in\mathbf{N}^*}A_i\right)=\lim_{n\to\infty}P\left(\bigcup_{i=1}^{n}A_i\right)=\lim_{n\to\infty}\left(1-\frac{1}{2!}+\cdots+(-1)^{n-1}\frac{1}{n!}\right)=\frac{1}{e}<\frac{1}{2}.\ \blacksquare$$

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