**The Mathematics Education into the 21st Century Project**
The Future of Mathematics Education
**Pod Tezniami, Ciechocinek, Poland**
**June 26th – July 1st, 2004**

# Mathematics – A Fresh Approach[1]

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*What can teachers of math do to promote high achievement in mathematics for their students?* This paper describes my attempts to improve mathematics instruction in a high-school where students show little interest in learning with not very interested in learning students.
One identified problem is the lack of motivation we, teachers, provide for abstracts concepts. "Why should I learn this concept?" is one of the questions I have often heard. Where could I use it? Is it applicable in our every-day life?
Nowadays, words as applying and applicable become more important than applied or applications. Students should learn how to apply mathematics; it is more important to learn how to apply mathematics than to learn applied mathematics. So, one of the questions I have tried to find an answer to was: *How should I teach mathematics so as to make students perceive it as useful?* Teaching mathematics by highlighting its usefulness should be a general concern of the teaching of mathematics. Students need to be helped to see and experience that mathematics is like a language into which we can translate many problems; we can solve problems with mathematical "instruments" and we can translate the result from the language of mathematics into our own "lay" language.
I am not saying that mathematics should be reduced to activities such as "telling" stories and translating them, making models or doing exclusively "pure" mathematics. Students should get a balanced "idea" of mathematics: as a means of solving problems and as a beautiful science with its own rules and laws, a science that is worth at the effort. The only problem is balance. Who will tell me that my mathematical approach is well balanced or not? I think that nobody will. As a teacher, I have to take this decision.
Another serious drawback of learning mathematics is that the students are not getting the sense that mathematics is a process. The compulsory curriculum and the traditional teaching methods used by many teachers fail to illustrate the way mathematicians actually think about and work on problems. Students are not able to understand that all the results in mathematics are obtained after many attempts, after struggling with one or more problems. They see mathematics as a set of given rules and laws that were established many years ago and that they are obliged to use now. They can see only a petrified structure with no creative trace.
Another question arose: *How should I teach mathematics and give students the possibility to understand the road that mathematicians cover to obtain solutions?* Following a common practice in the humanities, I have students read translated original texts. Original texts can enrich understanding of the roles played by cultural and mathematical surroundings in the invention of new mathematics. Through an appropriate selection of sources, students can appreciate immediate and long-term advances in their understanding of the clarity and elegance of concepts, techniques and notation.
Finally, I have identified another problem: reading and writing mathematical texts. Students have a lot of difficulty reading and understanding the original sources. Reading and writing activities can help students analyze, interpret and communicate mathematical ideas. These are skills needed to evaluate sources of information and the validity of the information itself, a key competency for 21st-century mathematically literate citizens. Many of the process skills needed for mathematics are similar to reading skills, and when taught together, they will reinforce each other. I asked myself, *How to motivate and engage students to speak, ask questions, write down their thoughts?* It's easy to do it when they are curious, exploring and engaged in their own mathematical inquiry.

[1] This paper is an adaptation of the article *Mathematics- A Fresh Perspective*, article that will be published in *Thinking Classroom*, a journal of the International Reading Association, in 2004.

My practical response to these questions was to introduce a new course for the students. I designed and delivered an elective course for the students in the 9th form (students aged 15 or 16): **History and application of mathematics**. By this elective course, I introduced them to a small part of the history of mathematics by outlining the connection between mathematics and other fields, and I gave students the opportunity to rediscover mathematical results, to *do* mathematics. I used a large variety of literature, including texts with mathematical theme (Keith Devlin's Mathematics – The New Golden Age; Simon Singh's Fermat's Last Theorem – The story of a riddle that confounded the world's greatest minds for 358 years; Ian Stewart's – Nature's Numbers – The Unreal Reality of Mathematical Imagination) in order to provide attractive information and to captivate students' interest.

The choice of the content was realized by taking into account the concepts the students already know from the 5th – 8th forms and the concepts students will study during the 9th form. We overlooked the part of history of mathematics, which deals with concepts the students do not know, and this part will be "an open door" for continuing the study of **History and application of mathematics** during the 10th-12th forms.

**The syllabus of the elective course: History and application of mathematics**

**The aims** of this elective course are:

- to discover information and ideas within a text
- to identify important information, to represent the information and to note givens and goals
- to seek relationships between given and goals, to identify solution steps
- to evaluate information and ideas for reflecting on the validity of the text
- to transfer mathematic acquisitions to other fields (biology, architecture, arts)
- to develop the social skills that support productive mathematical work with peers
- to experiment the mathematical creation

**Types of learning activities:**

- reading mathematical text/original writings, writing/re-writing mathematics, discussing mathematics;

***Note 1***: *Although most students "learn to read" during their first year of primary school, reading is a skill which continues to develop as the reading material becomes more sophisticated and as the expectations for the level of understanding increase. Mathematical reading (and for that matter, mathematical writing) is rarely expected, much less considered to be an important skill, or one which can be increased by practice and training. The activities and habits needed to learn from written mathematics are quite different from those involved in learning from a mathematics lecture or from those used in other types of text.*
*When we discuss about an original mathematical text, things become more complicated.*

***Note 2***: *In almost any advanced mathematics text, theorems, their proofs and motivation for them make up a significant portion of the text. The question then arises, how does one read and understand a theorem properly? What is important to know and remember about a theorem? A few questions to consider are:*

1) *What kind of theorem is this? (some possibilities are: a classification of some type of object, an equivalence of definitions, an implication between definitions…)*
2) *What is the content of this theorem?*
3) *Why are each of the hypotheses needed?*
4) *How does this theorem relate to others theorems?*
5) *What's the motivation for this theorem?*

***Note 3***: *Teachers can take advantage of students' innate wonder and inquisitiveness to develop language skills while learning mathematics concepts. Integrating literacy activities into mathematics classes helps clarify concepts and can make mathematics more meaningful and interesting. Writing a cinquian (a five-line unrhymed poem) encourages students to reflect on the topic/concept.*

***Note 4****: asking students to write 5 minute-essays about their problem-solving experiences or to articulate and defend their views about mathematics related issues provides opportunities to clarify the students' thinking and develop communication skills.*

*The students wrote frequently and about different aspects of the lessons: the mathematical details of the sources, their historical context, thoughts jotted in the throes of problem solving, and their own ideas about the process that creates mathematics. Writing time was scheduled and, somewhat to my surprise, the students wrote prolifically and generally quite well. This writing experience led to a more comprehensive view of the problems we studied as well as a much better grasp of mathematical details in their solutions. For example, after studying about the Golden ration the students had to*

- *write a 10 Minute-Essay: Explain, from your point of view, Johannes Kepler's (1571-1630) statement:*

*„Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel" or*

- *answer the question (in writing): Are there any advantages to the shape (spiral) of the clusters of seeds?*

***Note 5****: The ``discovery'' method assumes that students should discover mathematics for themselves. Hence, for each source I briefly provided the historical and mathematical context, alerted the students to any difficult points in the text, and then stood by to answer questions while they worked through the source in pairs. A wrap-up discussion let everyone share his or her understanding of the material, and any remaining difficulties were resolved. This method generated tremendous enthusiasm and a genuine sense of discovery. Strikingly, I saw that it also led to a deeper understanding of the sources than the lecture approach.*

- support for students' interaction with text by urging them to analyse the text, by encouraging them to "translate" the information into their own words/ mathematical symbols and to interpret the information

***Note****: Mathematics teachers can help all students increase their comprehension of mathematics texts by activating their prior knowledge through brainstorming, discussing the topic, asking questions and providing analogies.*

*Reading narrative mathematical texts are very good opportunities to develop students' critical thinking. When reading the texts, they have to identify the information they already know, the new information and the information they intend to deepen. In this way, they have to restate and to make an interpretation of the text, too.*

- tasks that can be carried out by strategies of cooperative learning / group work with established guidelines

***Note****: Cooperative learning helps teachers to develop and use critical thinking skills and teamwork of all members, promote positive relations among different groups, implement peer coaching, establish environments where academic accomplishments are valued, and even cooperatively manage schools.*

*Here are some strategies that can be used: STAD, Jigsaw II and Group Investigation. Student Teams – Achievement Divisions (STAD) – students with varying academic abilities are assigned to 4-5 member groups in order to study what has been initially taught by the teacher and to help each other reach his/her highest level of achievement. Students are then tested individually. Teams earn certificates or other recognition based on the degree to which all team members have progressed over their past records. Jigsaw II is used with narrative material. Each member is responsible for learning a specific part of a topic. After meeting with members of other groups who are "expert" in the same part, the "experts" return to their own groups and present their findings. Team members then are quizzed on all topics. Group Investigation – this is structured to emphasize higher order thinking skills such as* *analysis and evaluation. Students work to produce a group project, which they may have a hand in selecting.*

- study of original writings / sources

***Note**: By reading original sources, students are brought as close as possible to the experience of mathematical creation, without an intermediary interpreter. They see and feel the tenacity, the false starts and triumphs of its practitioners.*

*To supplement the study of original sources, we discussed relevant problem solving techniques, and students paired up to choose research topics. After that, the students had to re-word Proposition 47 from Euclid's Elements Book – Pythagoras' Theorem - and to prove it, they had to "discover" other ways to prove the same theorem: the Chinese proof, President J.A. Garfield proof, H.E. Dudeney proof, or to prove corollaries from Pythagoras' Theorem: the Arithmetic-Geometric means inequality, the cosine rule, Stewart's Theorem, medians, altitudes and Heron's formula or the general formula for obtaining Pythagorean numbers).*

**Instructional framework**

To deliver the elective course mentioned above, I organize teaching and learning in a framework for active learning and critical thinking. The model I use, known by its acronym **ERR**[2], begins with a phase called ***evocation***, in which students are encouraged to consider their assumptions about a topic, raise questions about it, are encouraged to set purposes for learning, and generally raise their curiosity about it.

The second phase is called ***realization of meaning,*** during which students inquire, examine and construct meaning. In the third phase of ***reflection,*** the students think back over what they have learned and compare it to their prior assumptions; they apply the learning to new situations, they question or debate the ideas, and they begin to reorder their thinking to accommodate what they have learned. I usually continue the lesson outside the classroom, with a fourth phase, called ***extension activity***, an activity that extends the ideas into further study of real-life applications. After learning about the Golden Ratio (Proposition 30 from Euclid's Elements Book VI), the students are asked to draw the logarithmic spiral and to find out the connection with the golden ratio, to find examples of logarithmic spiral in the natural world or to find the connection between geometry and architecture by explaining what Modulor means and by finding the sequence of values of the Modulor. There are many other possibilities: they can identify the Golden Section in Leonardo da Vinci's paintings: The Annunciation and Madonna with Child and Saints, or they can identify the Golden Section in the violin construction or in music: Mozart sonatas, Beethoven's Fifth.

**Content:**

1. History of mathematics. Sources of the history of mathematics
2. Natural numbers – the history of digits
3. The appearance of integers and rational numbers
4. Irrational numbers
5. Nature's Numbers
6. Algorithm – brief history and application on solving linear equation with one unknown
7. Famous problems: Pythagoras' Students, A Custom Duty, Diophantos' Epitaph, The School Master's Students
8. Pythagoras' Academy – academy, sect or secret political association
9. The Pythagorean Theorem
10. Perfect and amicable numbers
11. Pythagoras' statement "Everything is number"
12. The Golden Ratio
13. Fermat's Last Theorem
14. The evolution of the Number Theory

**Assessment methods and tools:**

[2] The ERR model was developed by the Reading and Writing for Critical Thinking Program

- Teacher observation
- Personal communication
- Projects + Students performances
- Essays
- Self evaluation
- Peer group assessment:
  - monitoring individual or group presentations
  group-work, communication, planning, organizing, presentation skills
  - monitoring project work in teams, complementary roles tend to be adopted
  - monitoring written assignments involving students in setting criteria
- Portfolio – most of these materials will become a part of the student's portfolio. Portfolios will have, at least, the following pieces: an essay written by the student on his/her expectations, a self evaluation sheet, an essay on the development of mathematical ideas, a self-report of learning by the student on the course (which will include answers to questions such as, What have you learnt? Why do you think that this course is important in your mathematics education? How will you be able to use the knowledge that you have gained in this course?)

I have been teaching this course for five years. I changed it a little bit each year because the students' assessment of the course gave me some new and interesting ideas. In this article I have presented the last form – the form I am using this academic year.

The end result, each year, has been a perception of mathematics different from the one students get from traditional courses. Students begin to view mathematics in a new way and they see themselves in a different relation to it. The students see mathematics as an evolving human effort, its theorems the result of human beings, people who struggled with the mysteries of the mathematical world, rather than an unmotivated, stony edifice of axioms and theorems handed down without human intervention. The theorems and the mathematical notions thus become humane.

An additional feature of the course is that value judgments need to be made: there are two geometries: the Euclidian one and the spherical one – which one is the "good" one? There are elegant proofs and confusing ones, there are plenty of mistakes and unsubstantiated assertions that need to be examined critically. Later follows the natural thinking that new mathematics is being created even today, quite a surprise to many students.

Here is one of the students' opinions about this elective course:

*"This course was for me like a breath of fresh air within the traditional mathematics I'm studying in the compulsory curriculum. I began to like mathematics and I obtained better grades at mathematics because I started to work on the exercises and problems. They are no more as difficult for me as they seemed to be." – R. H.*

The intention of this article has been to share with other teachers a different kind of approach to mathematics. The teacher's role is to engage students in learning mathematics and provide learning experiences through which students may construct understanding of the discipline for themselves. This elective course has helped me to carry out this hard task. The enthusiasm of the students during the lessons indicates that a humane and dynamic vision of mathematics, mathematical explorations in their own context and with their own motivation not only engage students in learning mathematics, but also enable them to think critically about the mathematical universe.

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