The Mathematics Education into the 21st Century Project
The Future of Mathematics Education
Pod Tezniami, Ciechocinek, Poland
June 26th – July 1st, 2004

BINARY DECIMAL NUMBERS AND DECIMAL NUMBERS OTHER THAN BASE TEN

Thomas Kim-wai YEUNG and Eric Kin-keung POON
tkwyeung@ied.edu.hk ; kkpoon@ied.edu.hk
(Mathematics Department, Hong Kong Institute of Education)

Abstract. *Most mathematics teachers are familiar with only base ten decimal numbers. Many of them may know the necessary conditions of when a rational fraction may be a finite decimal, an infinite pure recurring decimal or, an infinite mixed recurring decimal under base ten. However, what about decimal numbers with bases other than ten? Will the necessary conditions be the same? What do decimal numbers look like under base two? Since the Future of Mathematics Education links closely with computer structures which use the binary numeral system, it is both interesting and crucial that Mathematics teachers in the future familiarize themselves with decimal numbers under base two. This paper focuses on the representation of proper rational fractions by binary decimals. We narrow our scope to look at only proper fractions where a, b are positive integers with a<b and b not = 0 nor = 1. We would show and prove a theorem specifying the relationships between b and the base n = 2 which determine whether the representation will become either finite decimal, infinite pure recurring decimal or infinite mixed recurring decimal under base n = 2. Decimal numbers with bases other than ten or two will also be explored.*

1. Introduction

Most Primary Mathematics textbooks categorize decimal numbers into two distinct groups, namely finite decimal numbers and infinite decimal numbers. They then further separate infinite decimal numbers into recurring decimals and non-recurring decimals. This kind of categorization misleads students and teachers into erroneous notion that finite and infinite recurring decimals are two absolute distinct number types. In fact, finite and infinite recurring decimals are of the same number type called the rational numbers. Whether a number has finite or infinite recurring decimals depends solely on the base we are considering. A finite decimal under base ten can be an infinite recurring decimal under base two. For example 1/5=0.2000… is a finite decimal under base ten but 1/5 also=$0.\underline{0011}001100110011$…., an infinite recurring decimal under base two with period=0011. That is why some mathematicians would like to say that all finite decimals can be regarded as infinite recurring decimals with 0 repeating infinitely. An in-depth study of Binary decimals would enhance Mathematics teachers' understanding of decimal numbers in general.

2. Binary decimal numbers

Binary numbers uses only two numerals: 0 and 1 and each place value corresponds to a power of two. In decimal representation, 0.1 represents 2^-1=1/2 and 0.01 represents 2^-2=1/4, 0.001 represents 2^-3=1/8, 0.0001 represents 2^-4=1/16 and so on. Thus, 1/2 in binary decimal representation will be equal to 0.1, 1/4 in binary decimal will be 0.01 and 1/8 = 0.001 and 1/16 will be equal to 0.0001 etc…It is not difficult to see that for any positive integer n, 1/2^n will be represented by 0.0000…..1 with n places after the decimal and that they are all finite decimals.

Let's take a look at a list of proper fractions represented by Binary decimals:

1/2=0.1000…(finite)
1/3=$0.\underline{01}0101010101$… 2/3=$0.1\underline{01}010101010$…
1/4=0.01000…(finite)
1/5=$0.\underline{0011}00110011$… 2/5=$0.\underline{0110}01100110011$…
1/6=$0.0\underline{01}010101$… 3/5=$0.\underline{1001}10011001$…
1/7=$0.\underline{001}001001001$… 2/7=$0.\underline{010}01$…
1/8=0.001000…(finite) 3/7=$0.\underline{011}011$…
1/9=$0.\underline{000111}000111000111$…
1/10=$0.0\underline{0011}00110011$…
1/11=$0.\underline{0001011101}$…
1/12=$0.00\underline{01}010101$…

1/13=0.00010011101…
1/14=0.0001001001001…
1/15=0.000100010001…
1/16=0.0001000…(finite)
1/31=0.000010000100001…
1/32=0.00001000…(finite)
1/63=0.000001000001000001…
1/64=0.000001000…(finite)
1/127=0.000000100000010000001…
1/128=0.0000001000…(finite)
1/255=0.000000010000000100000001…
1/256=0.00000001000…(finite)
1/511=0.000000001000000001000000001000000001…
1/512=0.000000001000…(finite)
1/1023=0.0000000001…
…………….

A. From, the above list, we may notice a few conditions:

- Condition for Finite decimals: When the denominator b has only powers of 2's as factors, the fraction 1/b will be a finite binary decimal.
- Condition for Pure Recurring decimals: When the denominator b has only factors other than powers of 2s, that is, when the greatest common divisors, gcd(b, 2)=1, or, when b and 2 are relative prime, the fraction 1/b will be a pure recurring binary decimal. For example: 1/3, 1/5, 1/7, 1/9, 1/11, 1/13, 1/15, 1/31, etc… gcd(3,2)=1, gcd(5,2)=1, gcd(7,2)=1, etc…
- Condition for Mixed Recurring decimals: When b has factors consisting of both powers of 2's and other numbers not equal to 2 or powers of 2's, that is gcd(b,2)=2^n, the fraction 1/b will be a mixed recurring binary decimal. For example 1/6, 1/12, 1/14, 1/20, etc… We know that 6=2x3, 12=2x2x3, 14=2x7, 20=2x2x5.

B. Length of the period of recurring decimals:

To find the length of the recurring decimals, we need to look at the factors of (2^n – 1).

2^2 – 1 = 3
2^3 – 1 = 7
2^4 – 1 = 15 = 3x5
2^5 – 1 = 31
2^6 – 1 = 63 =7x9= 3x21=3x3x7
2^7 – 1 = 127
2^8 – 1 = 255 = 5x51 = 3x5x17
2^9 – 1 = 511
2^10 – 1 = 1023 = 3x11x31
2^11 – 1 = 2047 = 13x157
2^12 – 1 = 4095 = 5x13x3x3x7
…………

When b = a factor of 2^n – 1, n would be the period of the recurring binary decimal of 1/b or a/b (taking the lower n if the factor appears more than once in the above list). For example: when b=3, the period of 1/3 is 2, when b=7, the period of 1/7 is 3, when b=5, the period of 1/5 is 4, when b=31, the period of 1/13 is 5, when b=21, the period of 1/21 = 6, when b=127, the period of 1/127 is 7, etc…

C. Exercise for the Mathematics teachers:

A good exercise for the Primary Mathematics teachers would be to ask them how they would construct the recurring decimals of 1/17, 1/18, 1/20, 1/21 etc… If they can do so, it would show that they have a pretty good understanding of recurring decimals and the length of the periods of binary decimals. It would surely facilitate their understanding of decimal numbers of other bases.

3. Base ten decimals

Most people are familiar with decimal numbers base 10. We know that a proper fraction a/b will be a finite decimal number base 10 if b has factors consisting of powers of 2's and powers of 5's only. A fraction a/b will be a pure recurring decimal base 10 if b does not have 2 or 5 as factors. Fraction a/b will be a mixed recurring decimal base 10 if b consists of factors other than 2 or 5 and some 2 or 5 as factors. Why 2 and 5? This is because 2 and 5 are factors of 10 which is the base under consideration.

4. Base three decimals

Similarly, for base three decimals, a fraction a/b will be a finite decimal if b consists of only powers of three as factors. If b has only factors other than powers of three, a/b will be a pure recurring base three decimal. Fraction a/b will be a mixed recurring decimal if b has factors consisting of both powers of three and numbers not equal to three nor powers of three.

[Unless otherwise indicated, we'll from now on take a/b as a proper fraction in the most simplified form, a<b, and that a, b, n are positive integers greater than 0, and b not equal to 1. Q is a set of rational numbers.]

5. Theorems and Results

Theorem 1. Given that $a < b$, where a and b are positive integers and $(a, b) = 1$. $\frac{a}{b}$ is a pure recurring binary decimal if and only if $(b, 2) = 1$. Moreover, the length of the period of the recurring binary decimal would be the smallest positive integer n such that $b \mid 2^n - 1$.

Proof. Suppose $\frac{a}{b} = 0.\dot{a}_1 a_2 \cdots \dot{a}_t$, where $0 \le a_1, \cdots, a_t \le 1$, and t is the length of the period, then

$$\frac{a}{b} = a_1\, 2^{-1} + a_2\, 2^{-2} + \cdots + a_t\, 2^{-t} + \cdots$$

$$2^t\, \frac{a}{b} = a_1\, 2^{t-1} + a_2\, 2^{t-2} + \cdots + a_t + \frac{a}{b}$$

So, $a(2^t - 1) = bq$, where $q = a_1\, 2^{t-1} + a_2\, 2^{t-2} + \cdots + a_t \in Z$, hence $b \mid a(2^t - 1)$ and this implies $b \mid 2^t - 1$, so $(b, 2) = 1$.

The converse is similar and we skip for the reader. Now, suppose $b \mid 2^n - 1$, then $2^n = kb + 1$, i.e. $\frac{2^n}{b} = k + \frac{1}{b} > 1$. So, $(2^n - 1)\frac{a}{b} = kb \times \frac{a}{b} = ka = r$ (let). Since $0 < \frac{a}{b} < 1$, $0 < (2^n - 1)\frac{a}{b} < 2^n - 1$ and therefore, $0 < r < 2^n - 1$ and this implies the representation of r will become $r = r_1 r_2 \cdots r_n$, where $0 \le r_1, \cdots, r_n \le 1$. Since $2^n\, \frac{a}{b} = r + \frac{a}{b}$, $\frac{r}{2^n} = 0.r_1 r_2 \cdots r_n$, and $\frac{a}{b} = 0.r_1 r_2 \cdots r_n + \frac{1}{2^n}\frac{a}{b} = 0.r_1 r_2 \cdots r_n r_1 r_2 \cdots r_n + \frac{1}{2^{2n}}\frac{a}{b}$, continue the process, we have

$$\frac{a}{b} = 0.\dot{r}_1 r_2 \cdots \dot{r}_n .$$

Because t is the length of the period, we have $t \mid n$ and the proof is completed.

Theorem 2. Given that $a < b$, where a and b are positive integers and (a, b) =1. $\frac{a}{b}$ is a finite binary decimal if and only if $b = 2^n$ for some positive integer n. In this case, n is the length of the binary decimal.

Proof. If $b = 2^n$, and since $0 < \frac{a}{b} < 1$, we have $0 < a < 2^n$, hence

$a = 2^{n-1}a_1 + 2^{n-2}a_2 + \cdots + 2a_{n-1} + a_n$ for some $0 \le a_1, a_2, \cdots, a_n \le 1$ and $a_n \ne 0$.

So, $\frac{a}{b} = \frac{a}{2^n} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} = 0.a_1a_2 \cdots a_n$. In conclusion, it is a finite binary decimal and the length of the binary decimal is n.
The converse is obviously and hence the proof is completed.

Theorem 3. Given that $a < b$, where a and b are positive integers and (a, b) =1. $\frac{a}{b}$ is a mixed recurring binary decimal if and only if neither (b, 2) = 1 nor $b = 2^n$.

Remark. The proof can be easily obtained by the Theorem 1 and the Theorem 2.

6. Conclusion

Binary decimals is the most fundamental and the easiest decimals among decimals of all other bases. It is because 2 is the smallest prime number and is used as the base. Besides, it uses only two numerals: 0 and 1. Binary decimals should be taught alongside base ten decimals in all Primary schools and/or Secondary schools. Understanding binary decimals enhances understanding of base ten decimals and decimals of other bases. Students and teachers should find binary decimals fun to learn and work with.

References

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