PLENARIES

CIEAEM 71

Braga (Portugal)

July, 22 - 26 2019

CONNECTIONS AND UNDERSTANDING IN MATHEMATICS EDUCATION: MAKING SENSE OF A COMPLEX WORLD

CONNEXIONS ET COMPREHENSION DANS L'ENSEIGNEMENT DES MATHEMATIQUES: DONNER UN SENS A UN MONDE COMPLEXE

"Quaderni di Ricerca in Didattica (Mathematics)", Numero speciale n. 7, 2020
G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Making sense of probability in professional and everyday life

Carmen Batanero

University of Granada, Spain

E-mail: batanero@ugr.es

Abstract. Probability is today a strong branch of mathematics providing numerous models that can be applied in the many uncertain situations that surround us and guide decision making in different professional settings. However, although probability is included today in the school curricula from primary education, the time for teaching is scarce and students finish compulsory education with only routine knowledge of probability. Moreover, intuitions in probability often fail and a formal teaching does not help students overcome their reasoning biases. In this paper, I first analyze the different meanings of probability and its presence in the school curricula, then I present some examples of probabilistic decision making in different areas and finally suggest the need to link probability to every day and professional life to reinforce probabilistic reasoning in all the educational levels.

Résumé. Les probabilités sontt aujourd'hui une branche forte des mathématiques qui fournit de nombreux modèles pouvant être appliqués dans les nombreuses situations aléatoires qui nous entourent et qui guident la prise de décision dans beaucoup de contextes professionnels. Cependant, bien que les probabilités soient aujourd'hui incluses dans les programmes scolaires dès l'enseignement primaire, le temps d'enseignement est rare et les élèves terminent l'enseignement obligatoire avec seulement une connaissance routinière des probabilités. De plus, les intuitions en matière de probabilité échouent souvent et un enseignement formel n'aide pas les élèves ou les étudiants à surmonter leurs biais de raisonnement. Dans cet article, j'analyse d'abord les différentes significations des probabilités et leur présence dans les programmes scolaires, puis je présente quelques exemples de prise de décision probabiliste dans différents domaines et enfin je suggère la nécessité de lier les probabilités à la vie quotidienne et professionnelle pour renforcer le raisonnement probabiliste à tous les niveaux d'enseignement.

1. Introduction

Although the teaching of probability has been present in non-university curricula in the past decades, we find many suggestions to renew its teaching, with the aim of making it more experimental, so that students can be provided with a stochastic experience from childhood (e.g., CCSSI, 2010; MECD, 2014; 2015; NCTM 2000). These aims lead us to reflect on the nature of probability, and the purposes of its teaching in compulsory education

The teaching of probability in high or secondary school levels is mainly justified by the commitment for applying probability models when generalizing the findings of data analysis performed on samples to the populations from which these samples have been taken in the study of statistical inference (Batanero, & Borovcnik, 2016).

Another less frequently argued reason to justify the teaching of probability is the need to offer students tools to analyse the random phenomena, which surround us from birth. Due to the pervasiveness of randomness, students should overcome their deterministic thinking to become competent citizens in modern society, and when reaching adulthood, they have to accept the existence of fundamental chance in nature and in the world around us (Borovcnik, 2016). Consequently, they need to acquire strategies and ways of reasoning that help them in making

adequate decisions in those every day and professional situations where chance is present (Batanero, Chernoff, Engel, Lee, & Sanchez, 2016). Moreover, although chance is inherent in our lives and appears in multiple everyday situations or professional life (Bennet, 1999), intuitions in probability often deceive us and formal teaching is insufficient to overcome reasoning biases that can lead to incorrect decisions (Chernoff & Sriraman, 2014).
These reasons, which are not completely separate, should guide the content and methodology in the classroom, although some teachers focus only on the first goal, in assuming that the other aims are implicit in it (Gal, 2005). This belief may explain that most applications in the teaching of probability are reduced to games of chance, because the associate random experiments are simple and this context is highly motivating for the students. We agree that games of chance are powerful contexts, and, in fact, the first formal developments of probability were linked to problems related to these games (Batanero, Henry & Parzysz, 2005). However, it is important to offer the students a wider range of applications, so that probability does not appear to be a part of recreational mathematics. Many other different applications arise in almost all areas of human activity, such as, for example biology, medicine, risk analysis, education, management, climate or voting, and contribute to making sense of the different meanings of probability as discussed below.
In this paper, we first briefly analyse the meanings of probability presented in the school curricula and then describe examples of contexts that can be used to connect probability with everyday life and professional settings. These random situations of everyday life are usually more complex than the school problems presented in the teaching of probability, but are accessible for the students and help them make sense of probability and of the reasons why probability is useful to them. As a conclusion of our presentation, we suggest the interest of showing a wide range of applications of probability, especially in secondary education, and of strengthening the development of students' probabilistic reasoning to make them competent for real life.

2. Different meanings of probability

Probability is today a lively branch of mathematics in continuous growth; however, this area is still very young when compared with other parts of mathematics, such as for example, geometry, as can be observed when looking at the dates in which the Euclid's and the Kolmogorov's axioms were produced.
In spite of this short time elapsed since the mathematical study of probability started, there is not just only one definition of the concept. On the contrary, probability has received different interpretations since its emergence, which are related to the philosophical theories accepted in various historical periods. Even today, and in spite of having a satisfactory axiomatic system, there are still controversies over the interpretation of basic concepts and about their impact on the practice of statistics (Batanero, et al., 2005). Moreover, the progressive mathematical development of probability has been linked to a large number of paradoxes that show the disparity between intuition and conceptual development in this field (Borovcnik & Kapadia, 2014a; Székely, 1986).
Although the formal development of the probability appeared much later, intuitive ideas about chance emerged very early in history in many different cultures and were linked to problems related to setting fair betting in games of chance (Batanero, & Díaz, 2007; Bennet, 1999; Borovcnik & Kapadia, 2014b). These intuitive ideas also are common in children who use qualitative expressions in their games (such as terms "probable" or "unlikely") to express their degrees of belief in the occurrence of random events.
Hacking (1975) remarked that probability had a dual character since its emergence, as two facets are visible in the concept from the very beginning of its history: on the one hand, there is a statistical side of the concept, which is concerned with finding the stochastic rules of random processes, based on empirical data- On the other hand another epistemic side views probability as a personal degree of belief that can vary from one person to another. From these two basic meanings different theories of probability developed. Here we only discuss the definitions that are often used

in teaching. Other different meanings of probability are described in Batanero and Díaz (2007), Batanero, et al. (2005) or Borovcnik and Kapadia (2014a).

2.1. Classical meaning

The earlier developments in probability were linked to problems related to games of chance, and then, the initial definition of probability was based on the assumption that all the possible events in each random experiment were equally likely. Thus, in his *Liber de Ludo Aleae,* Cardano (1961/1663) suggested to compare the number of total possible results and the number of ways the favourable results can occur in order to make a fair bet in a game of chance. Pascal (1963/1654) in his correspondence with Fermat solved the problem of estimating the fair amount to be given to each player in an interrupted game by dividing the stakes among each player's chances. Pascal and Fermat assumed equiprobability of the elementary outcomes in a fair game as a criterion to estimate the probability of a compound event made up of these outcomes.

In the classical definition of probability, given by Abraham de Moivre (1967/1718) in *The Doctrine of Chances* and later refined by Laplace (1986/1814) in his *Essai Philosophique sur les Probabilités*, probability is the fraction of the number of favourable cases to a particular event divided by the number of all cases possible, whenever all the possible cases were equally likely.

This definition has been widely criticised since its publication, since the definition of probability was based on a subjective interpretation, associated with the need to judge the equipossibility of different outcomes. Moreover, this definition of probability cannot be applied to most natural phenomena where this assumption may not be valid., since although equiprobability is clear when throwing a die or playing another chance game, it is not acceptable in complex biological or human situations, such as for example in medicine or sociology.

2.2. Frequentist meaning

The convergence of the relative frequencies for the same event to a constant value after a large number of independent identical trials of a random experiment was observed by many different authors but it was Jacob Bernoulli (1987/1713) who proved a first version of the Law of Large Numbers that assures the convergence of these relative frequencies to the theoretical probabilities. Bernoulli also proposed to assign probabilities to real events, in applications different from games of chance, through a frequentist estimate. Intuitively this theorem can be stated as follows: when repeating the same experiment enough times, the probability that the distance between the observed frequency of one event and its probability p is smaller than a given value can approach 1 as closely as desired. Given that stabilized frequencies are observable, they can be considered as approximate physical measures of this probability (Batanero & Díaz, 2007).

This frequentist definition of probability was supported by mathematicians such as Von Mises, Gnedenko, Renyi and Kolmogorov and extended the range of applications enormously. However, a practical problem is that we never obtain the exact value of probability but only an estimation of probability that varies from one series of repetitions of experiments to another. It is also difficult to decide how many trials are needed to get a good estimation for the probability of an event. Moreover, this approach is not appropriate when it is not possible to repeat an experiment under exactly the same conditions. From an epistemological point of view another criticism of the frequentist definition of probability is the difficulty of confusing an abstract mathematical object (probability) with the empirical observed frequencies, which are experimentally obtained (Batanero et al. 2005).

2.3. Subjective meaning

Both in the classical and frequentist meanings, probability is an "objective" value that we assign to each event. However, the Bayes' theorem, published in 1763, proved that an initial (prior) probability could be revised in light of new available data and be transformed into a posterior probability. Consequently, the probability of the same event may be different for various people,

depending on each previous piece of information and then, probability could be as a subjective value. Following this interpretation, authors like Keynes (1921), Ramsey (1926) and de Finetti (1937) considered probability as a personal degree of belief that depends on a person's knowledge or experience in relation to a given event. Probability is then linked to a system of knowledge and is thus not necessarily the same for all people. In these situations the probability is not a tangible physical property - therefore objective of the events that affect us (such as weight, colour, surface, temperature) but a perception or degree of belief in the likelihood of the person who assigns the probability about the plausibility of occurrence of the event (which will or will not occur).
From this subjectivist viewpoint, probability can be applied to non-repeatable situations and again the field of applications of the probability theory was expanded in fields such as economy or medicine. Today, the Bayesian School assigns probabilities to uncertain events, even non-random phenomena, although controversy remains about the scientific status of results which depend on judgments that vary with the observer. However, the impact of the prior subjective assignment of probability diminishes by objective data (Batanero et al., 2005).

2.4. Axiomatic meaning

Throughout the 20th century, many mathematicians contributed to the development of the mathematical formalization of probability. Kolmogorov (1950/1933) derived an axiomatic theory, using Borel's view of probability as a measurable function. His theory, based on set and measure ideas was accepted by the different probability schools because, the mathematics of probability (classical, frequentist or subjective) may be included in his axiomatic. Despite the strong philosophical discussion on the foundations, the applications of probability to all sciences and sectors of human activity expanded very quickly. However, the interpretation of what a probability is would differ according to the perspective one adheres to and the discussion about the meanings of probability is still alive in different statistics school. Moreover, this abstract axiomatic meaning is structural, and responds to the need of organization and structuring of the remaining partial meanings of probability. Additionally, in real life other meanings often appear mixed in the same situation.

3. Probability in the school curricula

As analysed by Batanero and Díaz (2007), the differences between the various meanings of probability does not only affect the definition of the concept, but involves various concepts, procedures or properties. For example, the concept of relative frequency does not appear in the classical definition or the Bayes' s theorem is neither needed for computing a classical or a frequentist probability.
Given these differences, teachers should consider carefully which approach is preferable at different school levels or to introduce a given application of probability. Our suggestion is that we need to present a plural meaning of probability to the students throughout secondary and high school, so that they can progressively acquire the full meaning for the concept of probability.
When looking into curricular changes we observe that before 1970, the classical view of probability based on combinatorial calculus dominated the secondary school curriculum. Around the nineties it was complemented with the axiomatic approach, to present the students a direct application of set theory in the era of "modern mathematics". Both approaches reduced the applications of probability to games of chance, where the number of events in the sample space is small and the events are equiprobable. This reduction led many teachers to consider probability a part of mathematics with little value (Batanero et al, 2005).
Today we attend to a predominance of the frequentist view in the teaching, supported by computers or other electronic tools (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016). Simulations and experiments are used to introduce an intuitive version of the Law of Large Numbers that is used to define probability as the "limit" of relative frequency in a large number of trials. The subjective meaning of probability, which is widely used in applied statistics, professional and everyday

settings is only considered via conditional probability in some countries in the last years of high school. Learning goals for secondary school include the understanding and application of concepts such as random experiment, sample space, simple probability, conditional probability and independent events random variables and expected value, (CCSSI, 2010; MECD, 2015; NCTM, 2000). In high school, we also find compound events and probability, total probability, and Bayes' rule. In the GAISE document (Franklin et al, 2007), an introduction to the normal distribution as a model for sampling distributions, basic ideas of expected value, and random variation are also mentioned. In the Spanish curriculum for the social sciences branch of high school (MECD, 2015), besides the binomial and normal distributions the Central Limit Theorem and its implications for approximating the binomial by the normal distribution are also included.
All this content may be applied in many different contexts that go beyond games of chance so that students can connect probability with other school topics and with every day and professional contexts. Chance is present in everyday life in many contexts in which notions of uncertainty, risk and probability appear. Not just the professionals, but any person has to react to messages in which chance and risk appear, and make decisions that affect them, judge the relationship between events or make inferences and predictions (Gigerenzer, 2002).

4. Making sense of probability in everyday life

First, probability knowledge and reasoning are needed in everyday decision-making situations (e.g., taking an insurance policy, voting, evaluating risks of accident, interpreting coincidences, etc.). This probabilistic reasoning includes the ability to: a) Identify random events in nature, our everyday life and society; b) Analyse the possible outcomes, and the conditions of the situation; c) Select an adequate probability model for stochastic situations and explore possible outcomes from the model; and d) Interpret the results of working with the model in the real situation.
Many probability problems in everyday life are open or have more than one possible decision and both mathematical and extra mathematical factors are involved in their solution. For example, the possible utility of a decision, does not always coincide with their mathematical expectation in games of chance, where the mathematical expectation is negative for players who continue the game because of the usefulness of a possible but very unlikely price (Batanero, & Borovcnik, 2016). Moreover, in the decisions and judgments of probability in everyday life we are led by intuition that frequently deceives us and commit fallacies that are not usually corrected simply with a formal learning of probability (Shaughnessy, 1986). In the following sections we analyse some examples.

4.1. Insurance policy

Let us for example consider that a person buys a house or a new car and wishes to take out an insurance policy that covers the cost of reparation in case of accident or loss. Borovcnick (2016) suggests the following situation where students can investigate the convenience of taking out a full-coverage insurance contract (see also Batanero & Borovcnik, 2016 for a complete analysis).

Example 1. Exploring convenience of an insurance. A person is considering the convenience of taking an insurance policy to cover possible costs in case of accident with his new car. Let us consider only two possibilities: after a terrible accident the car is destroyed (and the price of a new car is 30.000 euros), or there is no accident. The cost of the insurance contract is 1000 euros. Should the person take the insurance policy?
When representing the data in Table 1, we realize that we need more information to be able to solve the problem. More precisely. we need to estimate the probability that an accident happens; this estimation is made by considering the past frequencies of accidents, driving skills and habits of the car owner, as well as possible accidents caused by other drivers and the state of the roads.

Table 1. Possible cost depending on decision taken

	Taking the insurance	Not taking the insurance
No accident	1,000	0
Total wreckage	1,000	30,000

Let us assume that the probability of an accident is 0.05. Then, the expected cost for the driver is 30,000x 05 + 0x0.95 = 1,500 Euros, so that with these data it is convenient to take the insurance. However, if the risk of accident is smaller for example, only 0.01 the best decision is not taking the policy. Moreover, other criteria often intervene in the decision, such as for example, that the owner needs the car for his work or that he cannot afford the 30000 euros in case of total wreckage.

4.2. Boy or girl?

The law of the large number assures us that, in the long run, the relative frequency for a given event converges to the theoretical probability for that event. While an informal version (as that presented in these lines) is understandable for most students, there is a tendency to expect convergence even in a short series of trials for an experiment. This wrong expectation (a strong belief in compulsive gamblers) is due to the representativeness heuristics (Kahneman & Frederick, 2002), according to which people expect the mean or the proportion in a sample to closely resemble the corresponding population parameters, even when the sample is small.

For example, if we know that a family have four children, we intuitively expect that two of them are boys and the other two are girls. However, the most frequent event is to have three boys and a girl or three girls and a boy. Let's x be the number of girls in a family of four children. Since we deal with a binomial distribution, where the number of trials is n=4 and the probability of success is p=.5 we can compute the probability $P(x=2) = 0{,}375$ that is smaller than $P(x=1) + P(x=3) = 0.5$.

In spite of this, when a woman with three boys gets pregnant once more, she is sure that the next baby would be a girl. She forgets that the outcome is independent one from the other (and the process of getting pregnant in successive occasions has no memory!). It is paradoxical that the odds to get a boy or a girl remain unchanged given that the person still has 1, 2, 3, … 7 boys.

5. Making sense of probability in professional contexts

Probabilistic situations are not only frequent in real life; the main applications of probability appear in the workplace. Below we describe some well-known examples.

5.1. The prosecutor fallacy

Two fundamental questions arise when a member of a jury is faced with evidence with a scientific basis in a court trial. First, should the jury member admit the evidence? If so, how should he or she assess such evidence (Friedman, 2017). A widely reported error in legal settings is the Prosecutor's Fallacy (identified by Thompson & Shuman, 1987) that consists of wrongly assuming a very small probability that a defendant is innocent, given that evidence with a very small probability of happening when the person is innocent was found in the crime scene. This is a fallacy of statistical reasoning, typically used by the prosecutor to argue for the guilt of a defendant during a criminal trial and is due to confusion between a conditional probability $P(A|B)$ and the transposed conditional $P(B|A)$. These probabilities may be in fact very different in value, and there is no commutative property when dealing with a conditional probability.

A very famous example of this fallacy was the case of Sally Clark, an English woman who was accused of murdering her two babies, after each of them died of infant sudden death in the few weeks of life (Byard, 2004). In the trial, the prosecutor argued that the chance of two children suffering sudden infant death syndrome was 1 in 73 million, by erroneously squaring 1 in 8500, which is the likelihood of a death in similar circumstances. Members of the British Royal

Statistical Society later issued a statement arguing that there was "no statistical basis" for this reasoning of the prosecutor, but Sally was imprisoned for three years.
Three years later a statistics professor performed a complementary probability study that proved that the possibility of two successive infant sudden deaths was far more common than assumed in the trial and Sally was released from prison, but anyway she never recovered fully from the experience. Many other similar cases were revised in England and some other women imprisoned were also found to be innocent afterwards.
Another famous example of the prosecutor fallacy happened in 1964, when an older woman, who was walking back home in a suburb of the city of Los Angeles, was assaulted by a young blonde with a ponytail. The young woman was seen shortly after in a yellow car driven by a black man with a beard and moustache. Los Angeles police arrested Janet Collins, blonde, who combed her hair with a ponytail and had a black friend with a beard and moustache, who owned a yellow car. The prosecutor argued that the probability of finding in Los Angeles a couple that had all the aforementioned characteristics was only 1/12000000, and this proved the guilt of the detainees. Fortunately, the defender, using statistical data on the frequencies of the characteristics of the couple (for example, frequency of yellow cars) proved that the probability of finding the least one more pair with the same characteristics in the city of Los Angeles was 0.1836, which cannot be considered a rare event. Joan Collins was acquitted for lack of evidence
Another example described by Thompson and Shuman (1987) is when a blood stain found in a crime scene coincided with the type of blood of a person accused. Assuming that this type of blood only happens in 1 person among 10 million people, then the prosecutor's fallacy consists of inferring from this that the probability that the person is innocent is only 1 in 10 million. If the blood coincidence is the only evidence and there are for example, 20 million people in the city, the probability that the blood is from the person in question is only 1/20, and of course, the murdered person may have come from another city, which still diminishes the probability.
The result is counterintuitive because we have difficulties in interpreting small probabilities. An event with small probability is not impossible and frequently happens if the experiment is repeated a large number of times. Suppose, for example, that a robbery has been committed and a sample of genetic material (such as blood) is found in the crime scene. This sample is compared with the data available in 20,000 cases of the police records and a match is found between the DNA of the sample and that of person A, whose data is contained in the records. Suppose the probability of finding a random person with this type of DNA is only 1 in 10,000. Can we consider that person A is to be blamed for the crime? Of course not! The probability that at least 20,000 cases appear at least one match with the sample DNA is quite high, exactly.

$$1-\left(1-\frac{1}{10000}\right)^{20000} \approx 86\%$$

Because of the reported errors in statistical reasoning happening in court, Friedman (2007) suggests that both the judicial system and the scientific establishment, including statisticians and teachers of statistics have a large role in explaining and in recommending the better use of evidence in the courts. And of course, teachers of probability can use these examples to correct the fallacy of the transposed conditional, which is the belief that a conditional probability is commutative.

5.2. Assessing risks in massive screening

Another well-known and rich context to study probability is medical diagnosis, where sometimes there is a confusion of a conditional probability and its transposed. In this context these probabilities can be substantially different, especially in cases of massive screening campaigns, where a whole population is subject to preventive medical tests (Gigerenzer, 2002).
Let us assume that a doctor suspects that a woman has breast cancer and recommends the patient to pass a given biometric test (for example a mammography). Suppose the test yields positive results in 95% of people with that illness (A) (test sensitivity) and in 5% of healthy people (not A) (test

specificity). Which is the probability that the person suffers from breast cancer if we know that the prevalence of this illness in the general population is 8/1000 and the test result is positive?

Table 2. Frequencies of women with and without cancer and different test results in 100000 women

	Positive result	Negative result	Total
Cancer	760	40	800
No cancer	4960	94240	99.200
Total	5720	94280	100.000

The situation can be better analysed if we consider natural (absolute) frequencies and a big number of women taking the test. In fact, in countries such as Spain every woman is recommended to pass a mammography every 1-2 years to prevent the disease, in what is known as massive screening. Lets assume that we have 100.000 women taking the test; the different possible events that may happen when considering the state of the woman and the test results are shown in Table 2. We have considered the following data:

- *Illness prevalence P(A)*= 0.008
- *Test sensitivity P(+|A)*= 0.95
- *Test specificity* P(- |not-A)= 0.95

From this table the following results may be computed:

- Probability that the person has the illness given a positive result of the test. $P(A|+) = \frac{P(A\cap+)}{P(+)} = \frac{760}{5720}$. Contrary to what many people expect this probability is smaller than the test specificity. The reason is the high number of healthy women that pass the screening and consequently, after a positive result of mammography if no other symptom is present the woman should require (and the doctors should recommend) new tests.
- Probability that the person has no disease given a negative result, that is, $\mathrm{P}(noA|-) = \frac{P(noA\cap-)}{P(-)} = \frac{94240}{94280}$= 99,9 and then a negative result is much more reliable than a positive result in a massive screening situation.

The person receiving a positive result of the test may be shocked, in not differentiating the conditional probabilities: "that the mammography is positive when the person has breast cancer" and "having breast cancer, then the mammography is positive. In Table 2 we observe that when we consider a group of 100,000 women and with the assumed proportion of cancer in the population, approximately 800 of these women would be ill and 760 of them would be detected on the mammogram. Another 4960 women would receive a positive result, even if they are healthy (false positive) In total we have 5760 positive mammograms in approximately 100,000 women, most of whom are actually healthy people.

The result may seem surprising and is due to the high rate of healthy women in the population who are tested annually, which leads to such a large number of false positives. On the other hand, even for women with cancer, the test is inconclusive, since approximately 10% of cases remain undetected (false negatives). The probabilities could be very different if the rate of patients in the population were higher (for example, if it were to detect the flu, or if the probability of false positive were lower (for example in older women).

Of course, the probability varies when dealing with a medical visit (not a massive screening) where the doctor also takes into account other considerations, such as age of patient, family history of cancer, habits (such as smoking, etc.) or another symptom. However, it is important that people learn to interpret medical diagnosis results and, when unconvinced, ask for a second test before taking an aggressive treatment, such as surgery.

6. Implications for teaching probability

The examples analysed drastically show the potential consequences of a poor intuition on probability and the relevance of many decisions we perform in uncertain situations. Following Fischbein (1975), the distinction between chance and determinism does not appear spontaneously and completely at the level of formal operations, because we are influenced by the cultural and educational traditions of modern society, which guide our thinking towards deterministic explanations. It is for this reason that research in probability education proves the prevalence of these erroneous intuitions (Batanero et al, 2016; Jones, 2005. Shaughnessy, 1992).

The few examples described in the paper highlight the relevance of probabilistic reasoning for every citizen. As suggested in Batanero et al. (2016), to adequately function in society, citizens need to overcome their deterministic thinking and accept the existence of chance in their lives. At the same time, they need to acquire strategies and ways of reasoning that help them in making adequate decisions in everyday and professional situations where chance is present. The probability education of all citizens is urgent, given the number of misconceptions and wrong heuristics described in probability education research.

Nevertheless, this education cannot be successful without an adequate preparation of teachers. A consequence of the philosophical ideas mixed with the different meanings of probability, the particular features of probabilistic reasoning, the students' misconceptions and difficulties, and the increasing variety of technological resources is that teachers need specific preparation to teach probability. Although school textbooks provide examples and teaching resources, teachers need additional support, as some texts present too narrow a view of probabilistic concepts or only one approach to probability (Batanero & Díaz, 2012).

First, teachers need adequate probabilistic knowledge that includes all the content they have to teach and an extended knowledge of probability that provides them with a perspective of the topic. Even if prospective teachers have a degree in mathematics, they have usually only studied theoretical probability and lack experience in designing investigations or simulations to work with students (Vásquez & Alsina, 2015). Moreover, recent research suggests that many prospective teachers share with their students' common biases in probabilistic reasoning or either are not aware of such biases (e.g., López-Martín, Batanero, & Gea, 2019; Vásquez, & Alsina, 2017).

However, even more, teachers need to increase their didactic knowledge to teach probability. In this sense we should take into account the different facets of didactic-mathematical knowledge describe by Godino. Giacomone, Batanero, and Font (2017):: epistemic facet (knowledge of mathematics), cognitive and affective facets (knowledge of students ways or reasoning, strategies in problem solving, affects and interests), mediational (tools and technology), interactional (ways of organised the classroom interaction) and ecological aspects (relationships of the topic with the curriculum, society, and other matters).

Acknowledgements

Project EDU2016-74848-P (MEC), Group FQM126 (Junta de Andalucía) and Group S36_17D-Research in Mathematics Education (Government of Aragón and European Social Fund)

References

Batanero, C., & Borovcnik, M. (2016). *Statistics and probability in high school*, Rotterdamn, The Netherlands: Sense Publishers.

Batanero, C., & Chernoff, E. J. (Eds.). (2018). *Teaching and learning stochastics: Advances in probability education research.* Springer.

Batanero, C., Chernoff, E., Engel, J. Lee, H., & Sánchez, E. (2016). *Research on teaching and learning probability.* New York: Springer. ICME-13. Topical Survey series.

Batanero, C., & Díaz, C. (2007). The meaning and understanding of mathematics. The case of Pçprobability. In K. Françoise (Ed.), *Philosophical dimensions in mathematics education* (pp. 107-127). Springer, Boston, MA.

Batanero, C., & Díaz, C. (2012). Training school teachers to teach probability: reflections and challenges. *Chilean Journal of Statistics, 3*(1), 3-13.
Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Ed.), *Exploring probability in school: challenges for teaching and learning* (p. 15–37). New York: Springer.
Bennett, D. J. (1999). *Randomness.* Cambridge, MA: Harvard University Press.
Bernoulli, Jacob (1987). *Ars conjectandi,* Rouen: IREM. (Original work published in 1713).
Borovcnik, M. (2016). Probabilistic thinking and probability literacy in the context of risk. *Educação Matemática Pesquisa, 18*(3), 1491–1516.
Borovcnik, M., & Kapadia, R. (2014a). A historical and philosophical perspective on probability. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: presenting plural perspectives* (p. 7-34). New York: Springer.
Borovcnik, M. & Kapadia, R. (2014b). From puzzles and paradoxes to concepts in probability. In E. J. Chernoff & B. Sriraman (Eds.), Probabilistic thinking: presenting plural perspectives (pp. 35-73). New York: Springer.
Byard, R. W. (2004). Unexpected infant death: lessons from the Sally Clark case. *Medical Journal of Australia, 181*(1), 52-54.
Cardano, G. (1961). *The book on games of chances*. New York: Holt, Rinehart and Winston. (Original work published in 1663).
Carnap, R. (1950). *Logical foundations of probability*, Chicago: University of Chicago Press.
Chernoff, E. J., & Sriraman B. (2014) (Eds.), Probabilistic thinking: presenting plural perspectives. New York: Springer.
Common Core State Standards Initiative, CCSSI (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. Online: www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.
Finetti, de B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institute Henri Poincaré, 7,* 1-68.
Fischbein (1975). The intuitive sources of probabilistic thinking in children. Dordrecht: Reidel.
Friedman, R. D. (2017). Controlling the jury-teaching function. *Seton Hall Law Review, 48*,815-825
Franklin, C., Kader. G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: A Pre-K-12 curriculum framework*. Alexandria, VA: American Statistical Association. Online: www.amstat.org/Education/gaise/.
Gal, I. (2005). Democratic access to probability: Issues of probability literacy. En G. A. Jones (Ed.), Exploring probability in school: Challenges for teaching and learning (pp. 39-64). Nueva York: Springer.
Gigerenzer, G. (2002). Reckoning with risk. Londres: Penguin Books.
Godino, J. D., Giacomone, B., Batanero, C., & Font, V. (2017). Enfoque ontosemiótico de los conocimientos y competencias del profesor de matemáticas. *Bolema, 31*(57), 90-113.
Hacking, I. (1975). *The emergence of probability*. Cambridge, MA: Cambridge University Press.
Jones, J. (Ed.) (2005). Exploring probability in schools: Challenges for teaching and learning. New York: Springer.
Kahneman, D., Slovic, S. P., Slovic, P., & Tversky, A. (Eds.). (1982). *Judgment under uncertainty: Heuristics and biases.* Cambridge university press.
Kahneman, D., & Frederick, S. (2002). Representativeness revisited: Attribute substitution in intuitive judgment. In T. Gilovich, D. Griffin, & D. Kahneman (Eds), *Heuristics and biases: The psychology of intuitive judgment,* (pp. 49,-81). New York: Cambridge University Pres
Keynes, J. M. (1921). *A treatise on probability.* New York: Macmillan.
Kolmogorov, A. (1950). *Foundations of probability's calculation.* New York: Chelsea Publishing Company (Original work, published in 1933).

Laplace P. S. (1986). *Ensayo filosófico sobre las probabilidades (*Philosophical essay on probability). Madrid: Alianza Editorial (Original work published in 1814).
López-Martín, M. M., Batanero, C., & Gea, M.M. (2019). ¿Conocen los futuros profesores los errores de sus estudiantes en la inferencia estadística? *Bolema: Boletim de Educação Matemática, 33*(64), 672-693.
Ministerio de Educación, Cultura y Deporte MECD MECD (2014). Real Decreto 126/2014, de 28 de febrero, por el que se establece el currículo básico de la Educación Primaria. Madrid: Autor.
Ministerio de Educación, Cultura y Deporte MECD (2015). *Real decreto 1105/2014, de 26 de diciembre, por el que se establece el currículo básico de la educación secundaria obligatoria y del bachillerato* (Royal decree establishing the structure of compulsory secondary-school and high-school curriculum). Madrid: Author.
Moivre, A. de (1967). *The doctrine of chances.* New York: Chelsea (Orig. work published in 1718).
National Council of Teachers of Mathematics, NCTM (2000). *Principles and standards for school mathematics.* Reston, VA: Author.
Pascal, B. (1963). *Correspondance avec Fermat* (Correspondence with Fermat). In Oeuvres Complètes (pp. 43-49). París: ed. Seuil. (Original work produced in 1654).
Ramsey, F. P. (1926). *Truth and probability.* In F. P. Ramsey (1931), The foundations of mathematics and other logical essays. London: Kegan, Paul.
Shaughnessy, J. M. (1992). Research in probability and statistics. En D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 465-494). New York: Macmillan.
Székely, G. J. (1986). Paradoxes in probability theory and mathematical statistics. Dordrecht, The Netherlands: Reidel.
Thompson, W. C., & Schumann, E. L. (1987). *Interpretation of statistical evidence in criminal trials.* Law and Human Behavior, 11(3), 167-187.
Vásquez, C., & Alsina, Á. (2015). Conocimiento didáctico-matemático del profesorado de educación primaria sobre probabilidad: diseño, construcción y validación de un instrumento de evaluación. Bolema-*Boletim de Educação Matemática, 23*(1), 681-703.
Vásquez, C., & Alsina, Á. (2017). Aproximación al conocimiento común del contenido para enseñar probabilidad desde el modelo del conocimiento didáctico-matemático. *Educación Matemática, 29(*3), 79-108.

Facing complexity with simplicity

Joaquin Gimenez

Universitat de Barcelona. Spain

E-mail: quimgimenez@ub.edu

Abstract. The presentation is a result of a reflection from 38 years of a teacher researcher. It is discussed a new challenge for Mathematics Teacher Education about the need of considering complexity framework: The need for improving the quality of epistemic analysis of a preservice teacher practice. It answers the question of such specific competence necessary in professional training programs for mathematics teachers to cope with the increasingly complex world challenges. It is used an example of tutoring a secondary school teacher when teaching volume for 13-14 years old students. Behind that, case study there is a discussion about the importance of facing complexity of mathematical objects and processes.

Résumé. Cette présentation est le fruit d'une réflexion d'un chercheur enseignant de 38 ans. Nous discutons d'un nouveau défi pour la formation des enseignants en mathématiques concernant la nécessité de prendre en compte le cadre de complexité : la nécessité d'améliorer la qualité de l'analyse épistémique d'une pratique de professeur de base. Il répond à la question de la compétence spécifique nécessaire dans les programmes de formation professionnelle des enseignants de mathématiques pour faire face aux défis mondiaux de plus en plus complexes. Il est utilisé comme exemple de tutorat d'un enseignant du secondaire lorsqu'on enseigne le volume à des élèves âgés de 13 à 14 ans. Derrière cela, se trouve une discussion sur l'importance de tenir compte de la complexité des objets et des processus mathématiques.

1. Introduction. About complexity of mathematics thinking.

Are the mathematical objects simple? *"Complexity thinking recognizes that many phenomena are inherently stable, but also acknowledges that such stability is in some ways illusory, arising in the differences of evolutionary pace between human thought and the subjects/objects of human thought"* (Morin, 2014). A complex system's capacity to maintain coherence is tied to the deep commonalities of its agents. Complex phenomena would appear with the ecology metaphor. The usual characteristics of a complex educational system are the following: self-organized, bottom-up emergent, short-range relationships, nested structure (or scale-free networks), organizationally closed (as inherently stable), structure determined, far-from-equilibrium (Davis & Sumara 2006).

According to this, we understand that there are a set of levels when interpreting teacher Education System (figure 1) from the understanding of training programs, the teaching competencies, the importance of Didactical analysis, and in particular epistemic analysis, and the understanding of mathematical objects and processes.

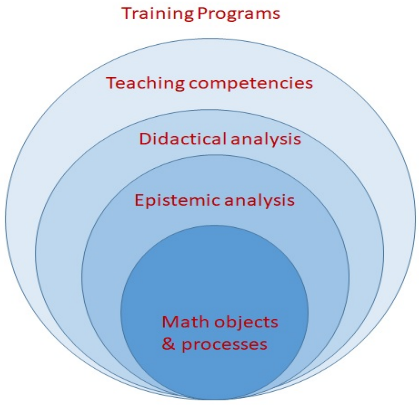


Figure 1. Systemic approach for a Teacher Training Program

Let's say in our case, an Interuniversity's Institutional Master's Training program in Catalonia since seven years ago. In such institutional background, we promote a set of contents and competencies by including the following settings:

A previous degree ensuring basic formation on mathematics, A psycho, socio, pedagogical educative discussion related to mathematics education. A revised mathematical discussion (problem solving, modeling, history of mathematics...). Such discussion should give elements for epistemic analysis.

It also includes a reflective practice in a Secondary school, including participation in school mathematics community. Elements of Didactical Analysis, assessment and suitability criteria, and a final reflection (Master professional thesis).

Epistemic analysis in a complex system.

Since cognitive 80's perspective, cultural aspects are first contextualised mathematical practices (pi). The different semantic meanings should be constructed connectively in a structured way over it, in a complex way. As I explained in my phD thesis, according Kieren's ideas using the example of rational numbers, I identified that the complexity of Mathematical objects seems to evoke more than a single unitary object, but a systemic complexity formed by parts or components build upon primary ideas.

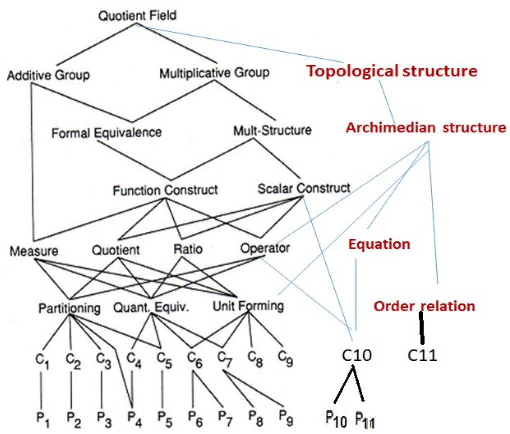


Figure 2. Rational numbers system as understood by Giménez (1991)

Mathematics Teacher Education should be understood as complex promotion of competencies, including creativity to understand epistemic complexity of mathematical objects. We should be creative if we know real phenomena associated to a math object and what mathematics can appear in real world experiences. The main aim is to introduce future teachers into a critical perspective by recognizing creative people, as Ferran Adrià did in a cooking environment. We assume that connections are in the heart of such a creative process. As Claudi Alsina said: "Neither microscope, nor telescope, just a math-scope" (Alsina, 1998).

In such a perspective, we assume a Didactical analysis approach following the model DCMM (Godino, Giacomone, Batanero & Font, 2017; Godino, Batanero & Font, 2019). There is a need of having suggestions for self- analysing mathematical practices. But, we do not have any Ratatouille who help us for self-control and reflection to say what is a "good mathematical practice". We need suggestions from outside ourselves. How to learn to manage and organize good practices, and what characteristics can we verify in them? How to help future teachers assess reflection as a generator of improvements in mathematical practices? An answer given in our research group is the recognition of a criteria of suitability or quality facing the school approach to the complexity of mathematical objects by using good practices.

A good practice is one that does not promote mathematical errors. Recognizing that mistakes have been made is a principle of improvement. But it also faces the ambiguities that arise from the metaphorical view of many mathematical ideas. The recent pragmatic proposal to talk about enriching mathematical processes, establishing connections between meanings seems to be important as well. And it is because it will be possible to imagine if a certain type of task puts the processes in evidence or not. And what contextual elements can favor better results. And how it serves to foster critical and creative people. Therefore, we decided to follow an epistemic initial problem by introducing the Institutional genesis of math knowledge is investigated in the ontosemiotic perspective (OSA) perspective by considering: a) to identify and categorize the problems-situations (phenomenology) needed to solve. b) To describe operational, discursive and normative practices during the resolution. c) To use a systemic set of configurations (definitions, processes, languages, propositions, arguments.

Now, almost 30 years after, we assume the need of being aware of Complexity of the mathematical objects by pragmatically observing relations among mathematical meanings and a systemic approach of relating mathematical configurations, as it is presented as an example for the average median (Font & Rondero, 2015). We assume that some suitability/quality criteria would help us to improve such didactical analysis. It means to consider Epistemic, Cognitive, Interactional, Mediational, Emotional and Ecological analysis using ontosemiotic approach. Specially considering the emergence of objects (Font, Godino & Gallardo, 2013).

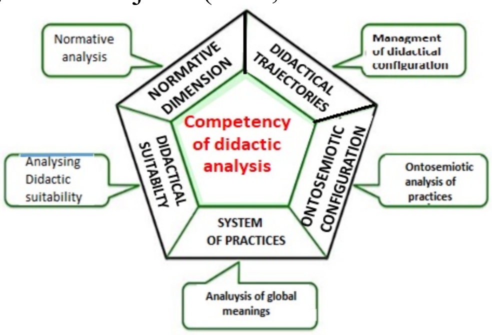


Figure 3. Integrated scheme of dimensions for didactical analysis in OSA perspective.

It means (as we see in figure 3), to consider an epistemic analysis, normative analysis, management of the didactical configuration, analysis of global meanings and didactical suitability, by observing the system of practices, didactical trajectories, ontosemiotic configuration and the normative dimension. There is also a need of a locally interpretation of such suitability criteria.

Therefore, in our theoretical perspective, we assume the suitability criteria as an optimisation problem for Teacher Education in the Ontosemiotic approach perspective (Giacomone, Godino & Batanero, 2017; Godino, Batanero & Font, 2019). The main new issue of such an approach is the need to a more complete analysis than the only cognitive perspective used analyzing p-primes knowledge.

2. Using an adequate training space for reflection

How to do such an analysis in a Teacher training program as we explained before? We assume a task design process (Giménez, Vanegas & Font, 2013). Searching for good reflective processes means identifying meanings of a math object, improving prospective teachers own practices. We use a path for the training process through several actions:

Planning the practice.

It is an important tutorial training phenomena during pre-service teacher formation by considering an a priori epistemic analysis to build a Study and Research Path (Barquero & Bosch, 2015). Valuing relevant contexts and adequate tasks. Would anyone say that a good practice is like a good vegetable stew? This is the metaphor that future professors think solves problems. For a while, Grandma's recipes were the patterns of good cooking. Good research has been thought to be good Wikipedia for teachers to learn. Nothing is further from reality.

In this part of planning the mathematical practice, it is needed to overcome a naif reflection about management and organization of the didactical units. Since the beginning of the Master formation, we leave the responsibility of the planning to the tutorial action. Observing the results of the final reflections, I intuitively feel the need to overcome the dependence of an experienced tutor when some final reflections are compared (Vanegas, 2012; Vanegas and Ferreres, 2015).

We see the importance of having a set of criteria to improve epistemic quality of the mathematics (Vanegas, Font & Giménez, 2016): Step 1. The need of being aware of mathematical errors. Step 2. Facing mathematical ambiguities. Step 3. Being aware of the need of including as much mathematical processes as possible. Step 4. Being aware of the connections among math meanings and limits of contextualisation watching the students' progress. And step 5) the need to identify the representativeness of mathematical meanings. It means the need of considering the hypothesis of a theoretical integrated systemic perspective for the meanings as a complex system.

We know that experienced teachers should have an initial proposal in which the mathematical activity is as enough rich of mathematical processes as possible. And it is one of the reasons that many teachers do not understand the proposal of mathematical "competential" practice as including the richness of processes. My hypothesis is that some of the criteria of epistemic suitability should be introduced since the beginning of planning. My proposal is now that to improve the awareness of reflecting about epistemic quality, is necessary an early start of the planning process including some suitability criteria to have a best organized sequence of mathematical tasks since a first version.

In the works of Sheckel & Font (2015); Breda, Font & Pino-Fan (2018), Font, Breda and Sheckel (2018) explain various examples such as the case of the Tales Theorem, or polynomials where teachers analyse didactic sequences they have made, in which there are gaps in recognition of said representativeness. In terms of analysing the quality of a teaching sequence, a criterion of suitability is that the set of practices implemented be as representative as possible of the system of practices that are the meaning of the object.

The reflection during implementation phase.

There is a need to have a reflection *during implementation* process. We think that it should pay attention to epistemic and cognitive suitability, interactive framework, and some normative aspects. After our recent experience, we also think that reflective process in this phase, should include reflection about *some representativeness of mathematic knowledge,* considering the need of an inter-related representative set of meanings and well connected (Giménez, Font & Vanegas, 2013; Breda, Font & Pino-Fan, 2018). The implemented practices should be a set as representative as possible of the system of practices considered about the meanings of the Mathematical object. In our experience, this is difficult in many cases of future teachers, because future teacher has no enough epistemic background. For many years, we assumed that it was important for future teachers to plan in a general way. But recently, we observe the need to introduce such reflection in our tutorial phase during the implementation.

The final reflection

The Final Master's Thesis is the last phase of reflection in which we propose the future teachers to analyze their practice by using suitability criteria. I will argue in the example of an analysis of a volume practice that a future teacher uses such scheme as a helpful element to analyze his practice.

3. A case study: Analyzing a sequence about the volume.

The aim of our proposal is to describe what happens in a teaching Program when suitability criteria (according OSA) are used to reflect about the complexity of math objects. Thus, to recognise the impact of these suitability criteria on the quality of epistemic analysis of future teacher practices. We'll use a case study about the consideration of complexity of volume with 13-14 years-old. A future teacher called Anthony is specially analysed.

When preparing his lesson plan, we discuss about the role of epistemic configuration to plan the practice. He identifies mathematical meanings even he knows the different contexts involved. But he is not fully aware of the need of connectedness for having representativeness of the most part of the meanings. And the difficulties to manage the use of time in his practice.

Observations before the teaching practice.

From helping decision making of the teacher to introducing phenomenological reflection as "break make-transformations" (Freudenthal 1983). First, the intention of such tutorial reflections is to outline the complexity of the concept, revealing the conceptions of children (Potari & Spiliotopoulou, 1996: 356-357). Then to connect to real associated phenomena, from Freudenthal perspective. After that, the future teacher decides to introduce different meanings, but any comment to this part of history.

During the tutorial moment we discuss with Anthony about some comments relating the famous article from Ricco and Vergnaud (1983) about the difficulties with discrete volume. We also discuss about the meanings considered by Piaget Inhelder y Szeminska (1970): (a) internal volume (amount of units of material in a body). (b) Occupied volume (amount of space occupied by a body related to others in the environment) (c) Displaced volume (when the object is submerged in a liquid). According to such discussion, we consider the future teacher adopted the following six meanings: M1 space occupied vs internal space observed against intuition; M2 Archimedian measurement as comparison; M3 Estimative computation by using Cavalieri principle and indivisible difference estimation vs continuous idea; M4 Flow measurement problems using equivalence relation vs capacity as linear dimension ; M5 Measuring as functional approach as opportunity to have benmarchs; M6 Integral definition not appropriated in this age.

During a first tutorial session, we discuss the need of taking decisions about the border of epistemic configuration that should be used in the lesson. Epistemic configuration means to explain the statement of the situation, definition adopted, representations, argumentations & reasoning. We also reflect a lot about how it appeared a historical evolution for the Volume of a pyramid, revealing powerful ideas from Liu Hui, arithmetisation to infinitesimal calculus.

Observing the written document reflection, Anthony think that his proposal allows the children to overcome difficulties found by Ricco and Vergnaud. We also found at first, the importance of following the following sequence is recognized: (1) emphasize the notion of volume as occupied space. (2) Contextualize conventional measures of cubic meter, cubic decimeter and cubic centimeter. (3) Distinguish volume area in discrete contexts. Assuming that in prismatic situations the volume is the product of the area of the base by the height. And assuming volume as an additive measure of AUB as sum of measures. (4) He recognize that the area can be minimized in situations of volume equivalence, by compaction. (5) He also establish the idea of volume as a relation to talk about the relationships between cube, pyramid, cylinder, and cone and sphere experimentally. And from there have elements to calculate volumes of these forms.

Reflections during the practice.

In this phase, the tutor attend to some classes to help the future teacher with its reflection and to help for better understanding of suitability criteria. Observing the classroom activity, we observe that the future teacher proposes (defines) that the volume is a space occupied by something. And this justifies talking about "*how big a cubic meter is and how important it is for children to see it*". The meanings are related to the problem fields and this with the context. Thus, in the pre-practice tutorial discussion, we discussed that Freudenthal pointed out that in volume ideas they occurred in contexts of liquids in the measurements of daily containers of water, milk juices, etc. Also in the consideration that prismatic volumes can be considered "sliced" bread.

Figure 4. Children observing the cubic meter

To do this, he contextualizes visually with sticks of 1m that take with their hands. Nevertheless, he immediately wants to intro of the conventional unit of measurement and for this he r connection with the volume of a milk carton, asking for "*how would fit in the cubic meter*".

The future teacher suggest the importance of relating the notic with the pluviometer, build by the students.

During the continuation, the future teacher proposes a good plan to analyze the computation of the volume, by using slices of bread, and to compare the possible equal volume with different positions. We can say that many of the future teachers did good planning, but we cannot see in their documents the same level of being aware about the complexity of the mathematical objects. Anthony prepare a good set of tasks, even without recognising the need of more connections among mathematical meanings. He is worried about an epistemic perspective of Mathematical practice in school (not common). He insist on the difference between discrete measurement of cubes and continuous measurement of sphere related to the need of approximation.

Immediate reflections after the classroom observation.

After watching the practice, the Trainer suggested some proposals to make more connections and the need to produce images for improving understandings. The tutor suggest for instance to include in a new practice. How many litters of liquid we eliminate when sports transpirations? Do you know about the formula for estimating the amount of blood in our body? He also suggest to contextualize more and to include a functional approach for volume, to see the variables influencing its computation.

Reflections at the final Master thesis.

Master's thesis, one month after give us opportunities to observe the influence of using suitability criteria. In Anthony's planning, there are a set of previous interesting elements that we have not observed in other students: a) it seeks to confront the complexity of the connection between the idea of space and form (with its dimensionality) and volume. b) Seeks that students face the complexity of the real world, distinguishing between physical properties of shapes, materials, etc. and the form as part of the space that has a volume. c) He has an epistemic (uncommon) concern for recognizing the mathematical as different from the physical. And mathematically, it identifies the difference between the discrete measurement of the cube count of the continuous measurement of certain objects such as the sphere, which allows an approximate and definable vision with real numbers. d) It is clear that the volume is linearized when considered as a capacity in the case of liquids. e) Try to integrate at least two meanings of volume. An initial concept of volume as a place, which allows us to talk about equivalence and the idea of volume as a measure that expresses a relationship.

From the analysis, we observe that the future teacher Anthony assume many of the suitability criteria proposed, mainly on epistemic reflection (Giménez, Font & Vanegas, 2013). The ***consideration of representativeness*** with certain difficulties when applying the tasks in the classroom. He uses a counting discrete approach to try to solve the problem of trilinearity, but does not relate contextual examples of re-shaping a volume in continuous context. After having discussed in the course the notion of representativeness, he recognized the mistake, and propose a way to solve it.

This future teacher considers ***the need to connect functional approach with direct measurement meaning*** in the discrete case. At that time, he proposes to improve this by using a geogebra mediation to reveal the use of variables when computing the volume of a box in a discrete case. He thinks that it can help more than the simple table he used in the class, to see what happens for a fixed volume if you change one separated variable. He says "then it is possible to visualize the invers proportional relation". Let us see what he said about ***richness of processes***. For instance, *"**Problem solving** articulated the whole experience. A good part of concepts arose as a necessity to solve situations, and it was only formalized once it had been discovered ".**Trial and error** processes were especially strengthened, since in many activities systematic explorations were included by means of physical manipulations (construct, touch, see).* ***"The argumentation…****Several problems (orthohedres cut, with the same volume, 3D constructions with the same views) asked / valued all the possible solutions, as well as the explanation of the systematization strategies that they used."* Around the equivalent volume **different intramathematic connections were worked on.** *"The problem of finding all possible orthohedres of equivalent volume required the factorization of integers as a strategy of systematization, connecting geometry, measurement and numbering".*

Anthony also introduced a deep reflection by analysing **epistemic "structureness"** that we see as a flavour of the recognition of contributing to a complexity of mathematical structure. *He said… "My new proposals introduce improvement in the sense of reinforcing proportional relations among volumes from a double side: (1) Direct measurement as unidimensional comparison. At the same time, the inverse proportionality between the size of unit and the corresponding measurement will be studied from the change of units. For a better consolidation, we will use as well the conventional as arbitrary units. (2) Facing the indirect measurement with the change of dimensionality. Thus the equivalence of volume will be presented in the context of displaced volume in recipes with different dimensions and corresponding vertical graduations, having a different perspective of traditional change of units".*

4. Final remarks.

The examples presented in the specific Anthony's case study show that Prospective Teacher fully appropriate himself the suitability criteria. He even argued in some conversation "*It seems to be a great idea for reflecting about the work done*".

According the observations, we see the influence of the future teacher using the suitability criteria when analysing their practice. Almost every future teacher considers to overcome the possible mathematical errors and ambiguities; richness of processes, their relevance and systemic perspective. Anthony and some others assume the need of connectedness among contexts used. It is also found the need for representativeness. The case of Anthony is special, because he made explicit the mathematical meanings as representative of the complexity of math notion. We finally assume that just a small number of future teachers also introduce a reflection about the structure of the mathematical object identifying the complexity of mathematical object. Nevertheless, definitions and arguments of the configuration are not completely detailed in his reflection.

The difficulties of future teachers in solving situations about the idea of volume are known. In many cases, teachers say that the main difficulty with volume is the problem of changing measures. But the difficulties that arise from the daring in facing complexity are less known. When this occurs as the case of Anthony partial meanings are discussed, and therefore it is more possible that complexity is considered. We finally interpret what Einstein said: EVERYTHING SHOULD BE AS SIMPLE AS POSSIBLE BUT NOT SIMPLER!

Acknowledgment

These results are partially granted by the project PGC2018-098603-B-I00 from the Ministry of Economy and Competitivity of Spain.

References

Alsina, C. (1998). Neither a microscope nor a telescope, just a mathscope, P. Galbraith, W. Blum, F Booker, ID Huntley and M. Niss (Eds.) *Mathematical Modeling, Teaching and assessment in a Technology rich world* Chichester: Hordword. Albion Publishing Ltd.

Barquero, B., Bosch, M. (2015). Didactic engineering as a research methodology: from fundamental situations to study y and research paths. In A. Watson & M. Ohtani (Eds.) Task *Design in Mathematics Education. An ICMI study 22* (pp. 249-272). Springer International Publishing.

Breda, A., Font, V., Pino-Fan, L. R., (2018). Criterios valorativos y normativos en la Didáctica de las Matemáticas: el caso del constructo idoneidad didáctica, *Bolema*, 32 (60) 255-278

Davis, B. & Sumara, D.J. (2006) *Complexity and Education: Inquiries into Learning, Teaching, and Research*. Psychology Press, Philadelphia.

Font, V.; Godino, J. D. y Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, 82, 97-124.

Font, V., Breda, A., & Seckel, M. J. (2017). Algunas implicaciones didácticas derivadas de la complejidad de los objetos matemáticos cuando estos se aplican a distintos contextos. *Revista Brasileira de Ensino de Ciência e Tecnologia,* 10(2), 1-23.

Font, V. & Rondero, C (2015). Articulación de la complejidad matemática de la media aritmética. *Enseñanza de las Ciencias,* 33(2), 29-49.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*, Reidel Pub. Co.

Gimenez, J. (1991). *Innovación metodológica del número racional positivo*. Tesis doctoral no publicada. Universitat Autònoma de Barcelona.

Giménez, J.; Font, V. & Vanegas, Y. (2013). Designing Professional Tasks for Didactical Analysis as a research process. In: C. Margolinas (Ed.), *Task Design in Mathematics Education. Proceedings of ICMI Study 22*: Oxford,

Godino, J. D., Giacomone, B., Batanero, C., & Font, V. (2017). Enfoque ontosemiótico de los conocimientos y competencias del profesor de matemáticas. *Bolema*, 31(57), 90-113.

Godino, J. D., Batanero, C. y Font, V. (2019). The onto-semiotic approach: implications for the prescriptive character of didactics. *For the learning of mathematics* 39 (1) 37-42.

Morin, E. (2010). Discours de monsieur Edgar Morin. En D. Ramirez; J. Coello. *Solemne investidura com a Doctor Honoris Causa del senyor Edgar Morin.* Barcelona, Espanya (pp. 53-100). Publicacions i Edicions de la Universitat de Barcelona

Piaget, J., Inhelder, B. y Szeminska, A. (1970). *The child's conception of geometry.* Londres, GB: Routledge and Kegan Paul.

Pino-Fan, L., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education*, 21 (1), 63-94.

Potari, D. y Spiliotopoulou, V. (1996). Children's approaches to the concept of volume. *Science Education* 80 (3): 341-360

Ricco, G., Vergnaud, G. (1983). "Représentation du volume et arithmetisation. Entretiens individuels avec des élèves de 11 a 15 ans". *Recherches en Didactique des Mathématiques*, 4 (1), 27–69

Seckel. M. J. y Font, V. (2015). Competencia de reflexión en la formación inicial de profesores de matemática en Chile. *Praxis Educacional*, 11 (19), 55-75.

Vanegas, Y (2012) El papel del trabajo final de máster en la formación del profesorado de matemáticas. *Uno Revista de Didáctica de las Matemáticas* 61, 76-85 |

Vanegas, Y. & Ferreres, S. (2015) Uso de Criterios de Calidad en la Reflexión Sobre la Práctica de Los Futuros Profesores de Secundaria de Matemáticas *Procedia - Social and Behavioral Sciences* 196:219-225.

Vanegas, Y; Font, V; Giménez, J. (2015) How future teachers improve epistemic quality of their own mathematical practices. CERME 9 - *Ninth Congress of the European Society for Research in Mathematics Education*, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp. 2937-2943.

Indigenous knowledges re-evaluating mathematics and mathematics education

Kay Owens

Charles Sturt University, Locked Bag 49, Dubbo, NSW 2830, Australia

E-mail: kowens@csu.edu.au

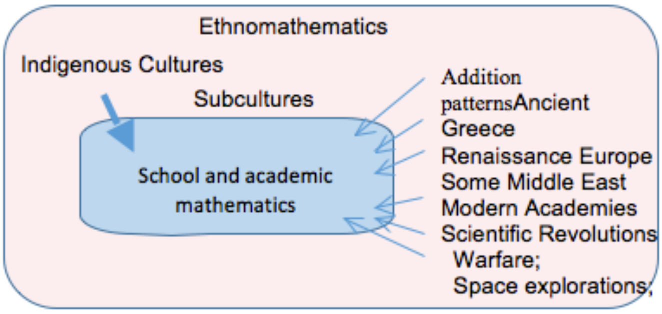
Abstract. From experiences and research in Papua New Guinea, three key theses can be established building on Indigenous mathematical knowledges. First the importance of visual and spatial aspects of mathematical reasoning. Physical and visual comparisons, halving and using ratios are involved in reasoning about estimations of measures of attributes of objects. Viewing and feeling in your body have become major ways of reasoning in Indigenous cultures. Nevertheless, most school curriculum begin with counting rather than looking at the visuospatial attributes of size. Examples of practice indicate that many alternative experiences could be provided in early education and be a way of making mathematics real for older students. Second, studies of counting systems establish the longevity of counting in New Guinea, Melanesia and the Pacific from 10, may be 40, thousand years ago – a little told fact in the histories of number. Recognition of the composite groupings in these systems assists the conceptualisation of number. Finally, a caution on curriculum: post Europe's colonialism, neo-colonial attitudes and aid advisers perpetuate a lack of recognition of Indigenous mathematical knowledges. 01A07, 01A13, 11-03

Résumé. D'après des expériences et des recherches menées en Papua New Guinea, trois thèses clés peuvent être établiées en s'appuyant sur les connaissances mathématiques autochtones. Premièrement, l'importance des aspects visuels et spatiaux du raisonnement mathématique. Les comparaisons physiques et visuelles, les rapports de moitié et d'utilisation sont impliqués dans le raisonnement relatif aux estimations de mesures d'attributs d'objets. Voir et ressentir dans votre corps sont devenus des moyens de raisonnement majeurs dans les cultures autochtones. Néanmoins, la plupart des programmes scolaires commencent avec le calcul, plutôt q'avec les attributs visuospatiaux de la taille. Des exemples tirés de pratique indiquent que de nombreuses expériences alternatives pourraient être proposées dans le cadre de l'éducation préscolaire et constitueraient un moyen de rendre les mathématiques réelles pour les étudiants plus âgés. Deuxièmement, des études sur les systèmes de dénombrement établissent la longévité du dénombrement en Nouvelle-Guinée, en Mélanésie et dans le Pacifique à partir de 10 000 ans, peut-être 40 000, ans - un fait peu connu dans l'histoire du nombre. La reconnaissance des groupements composites dans ces systèmes facilite la conceptualisation du nombre. Enfin, une mise en garde sur le curriculum: le colonialisme post-européen, les attitudes néocoloniales et les conseillers d'aide perpétuent un manque de reconnaissance des connaissances mathématiques d'Autochtones. 01A07, 01A13, 11-03

1. Introduction

This paper tells my story as a way of providing evidence of how Indigenous knowledges can transform one's perspective on mathematics and mathematics education. It also shows how learning occurs from this perspective and how education and curriculum can change. My understanding of mathematics has been heavily influenced by the enrichment of living in Papua New Guinea (PNG) for 15 years, teaching mathematics at the PNG University of Technology and teaching health education and education at Balob Teachers College followed by a further 30 years of research with my colleagues in PNG. This knowledge is strengthened by other ethnomathematics research studies and living in a rural city in colonised Australia on Wiradjuri land with Aboriginal Elders and colleagues sharing their understandings of *yindyamara winhanganha*, the wisdom of respectfully knowing, considered, going slowly, listening, valuing, caring, learning how to live well in a world worth living in (understanding of wisdom gifted by the Elders to Charles Sturt University).

There are three important theses that my colleagues have helped me to develop through Indigenous mathematical knowledges. These include the importance of visual and spatial aspects of mathematical reasoning; a new history of number; and a clearer perspective on PNG colonial and neo-colonial mathematics education. Summarising this research establishes that ethnomathematics should influence pedagogy and curriculum worldwide, in multicultural societies and classrooms, and Indigenous societies. Figure 1 represents an arguable ideal that results from recognising and valuing ethnomathematics or as D'Ambrosio has said "the backbone of mathematics" (2008)[1].



Source.Owens (2016)

Figure 1. Ethnomathematics incorporates other mathematics.

Papua New Guinea's main island has many steep valleys and large relatively fertile upland valleys in the highlands of the mainland. It has a coastal fringe with coral reefs and some wide lowland river valleys and sago swamps. It is half of the island of New Guinea and hundreds of large and small islands. It is north of Australia (who held it as Trust Territories through two World Wars), south of the Philippines, and east of Indonesia. Papua New Guineans are Melanesian but there are 850 languages and cultures. There are several language phyla of the older languages (Papuan or Non-Austronesian) and then there are Austronesian languages with many clusters, mainly around the coast in PNG as well as in Island Melanesia. Different cultural groups and families within these groups own the land and have close connections to the land but there was considerable trade between groups before colonisation. This ecological background impacts on people's mathematical activities. PNG languages and cultures are still very much intact although a few languages are endangered or quickly changing (see Figure 2 for a glimpse).

Port Moresby is the capital and it sits on a small parcel of land so has many high rise buildings, and many self-help housing settlements of people from all over PNG. Despite PNG's wealth (mines, coffee etc) and considerable aid (mostly Australian, Japanese and New Zealand Aotearoa), many PNG people live in poverty in towns alongside profitable enterprises or as subsistence farmers hunting, fishing and gathering foods like sago, fruits and nuts and managing cash crops such as coffee and copra. Recent environmental damages have occurred through climate change, under-regulated fishing, forestry, mining, and oil palm. Its population since Independence in 1975 has doubled creating many problems for education such as a lack of schools, large class sizes, limited or no materials, access to salaries, difficulties for management, lack of trained teachers, and inadequate professional development although some is amazing such as the early literacy courses run by SIL, our elementary project though too small, or the planned inservices of the mathematics association.

[1] Keynote address at International Congress for Mathematics Education, ICME11, Copenhagen, 2008.

Figure 2. Diversity of cultures, foods, handcrafted earthen cooking bowls, homes, and dress.

Infrastructure is limited for medical services, sewerage, telephone lines, running water, roads, and electricity except in towns and some more remote settlements and boarding schools. High fuel costs for dinghies; lack of teacher education for the elementary school teachers, and remoteness create some challenges for teachers and education. Its contact with Europeans and colonisation began around 1880s and continued as Territories through the World Wars until 1975 Independence when Department heads were Papua New Guineans and nearly all parliamentarians were of PNG cultural background. There are very few expatriates (mostly aid or church workers) or naturalised citizens (usually long-term business men), about 20 000 refugees from West Papua[2] (also Melanesian), and virtually no migrants (recently new Chinese have been entering and taking over trade stores). So in terms of colonisation, PNG's cultures have not been as affected as in other countries, thanks to the remoteness, the coral reefs, rugged terrain, lack of large edifices, and the independence of each of many language group.

2. Visuospatial Reasoning

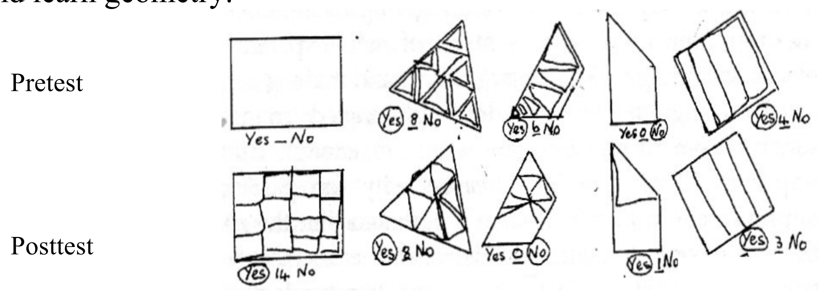
This study began with my observations and yarns in PNG in many villages when I went with my students for teaching practices or we went hiking into the villages. Colleagues like Geoff Smith and Glen Lean also discussed their research about the counting systems and I was able to read the many papers written about spatial research carried out in PNG (Bishop, 1978; Lean & Clements, 1981; Saunderson, 1973) while I was teaching mathematics for Architecture students and others at the PNG University of Technology. Eventually over the 45 years, I stayed for a few hours to three weeks in 60 villages across 14 Provinces, and each one left an impression of the people's rich culture including language and relationships between people and the land, their material culture, and their ways of thinking including cultural mathematics. In many cases, friendships with community members continued for many years. In many places, with permission, I took photographs and video of activities involving mathematics and discussed mathematical thinking with the people. I acknowledge the knowledge and sharing of the people I met and knew (know) well in PNG.

Returning to Australia, I carried out a mixed method, quasiexperimental study of spatial thinking during geometry problem solving by students in Grades 2 and 4, from three schools in multicultural low socioeconomic areas of South Western Sydney, a school in Western Sydney, and a town school in PNG. The students improved on the delayed posttest when compared to the matched control group who undertook number problems and pretest scores were taken into account ($n=179$, $F=5.07$, $p=.02$). The videorecorded episodes of groups in these classes together with interviews with immediate retrospective stimulated recall by students undertaking these geometry problems were analysed. It was a grounded theory study but it was influenced by Presmeg (1986) and Goldin (1987) in particular. Based on codes developed from the interview data in conjunction with the videos, each videod segment of learning was coded with the apparent cognitive processing and the

[2] The western half of the mainland and islands occupied by Indonesia; the name is used by refugees.

interaction of these processes. My analysis of this work indicated the importance of visual imagery along with attention (noticing, influences of past experiences, the problem, arrangement of the materials, and comments from others), heuristics such as what the problem was and what they could try, concepts and affective processes. Influenced by the concrete materials and comments of others, the cognitive processing resulted in responsiveness – doing and saying – which in turn influenced the position of concrete materials and what teachers and peers might next do or say. Thus the problem solving continued and their imagery and concepts developed (Owens, 1994, 1996; Owens & Clements, 1998). The model that I developed was extended as my research in Papua New Guinea continued (see figure 7 in this paper).

One aspect of this research concerned the developing visualisation of tiling of rectangular or other areas with square, rectangular or triangular tiles and how children began to structure their visualising (Owens & Outhred, 1996). Figure 3 shows a child's responses to the paper-and-pencil items of my test: covering a rectangle with a given square tile, an equilateral triangle and trapezium with equilateral triangles, then another trapezium with a right-angled isosceles triangle and the last square with rectangles following eight investigations such as making a pattern block shape larger with the same pattern block or covering tangram shapes with smaller tangram shapes. Noticeably, as (Outhred, 1993) also found, the students moved from single tiles to noting structure. Subsequently, a professional learning package was prepared for teachers on spatial reasoning and good teaching practices for problem solving learning that encouraged children to investigate, visualise and learn geometry.



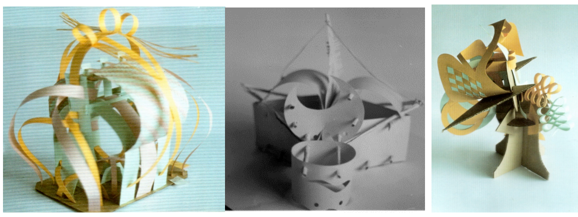
(Source. Owens & Outhred, 1996, p. 36)

Figure 3. Students' visualising tiling rectangles and other shapes.

This research influenced the NSW[3] Department of Education professional development program undertaken by Peter Gould, Diane McPhail and several teachers to establish an approach to teaching space and geometry for primary schools. We developed a series of learning experiences that focused on investigating and visualising, and describing and classifying. In particular, we used terminology familiar to teachers in early number to suggest at various times children will use differing kinds of visual imagery: emerging, perceptual, pictorial imagery, pattern and dynamic imagery and efficient imagery. We used video materials explaining the ideas, exemplar teaching in classrooms, and children undertaking assessment tasks demonstrating the various visual imagery approaches. Trial schools were a representative sample of schools supported by consultants or in-school facilitators, the latter providing the best support when the training packages were utilised (Owens, 2004; Owens, McPhail, & Reddacliff, 2003).

We continued to visit PNG for research with colleagues. One study was with Architecture students whose first design project was heavily influenced by their cultural experiences (Owens, 1999). Figure 4 shows three of the designs. The use of mathematical shapes together with canoes, ways of joining objects, recognition of space and other foundational cultural ways with weaving, decoration, and curves is evident in their paper-cardboard sculptures. However, there was a challenge when they said that "we are PNGian, we will be good architects". I wondered if students would say the same about mathematics.

[3] New South Wales is a state of Australia with Sydney as its capital.



(Source: Owens (1999, pp. 291-292)

Figure 4. Paper only sculptures reflecting imagery associated with PNG culture.

Wilfred Kaleva supported by Rex Matang[4] at the University of Goroka (UoG) were already studying ethnomathematics and together we developed a study of measurement practices using a questionnaire given to hundreds of tertiary students using intranet facilities, 16 interviews of staff and students, focus groups with students to check our developing understandings, and visits to villages in three different areas (coast, highlands, and coastal valleys) where the co-researchers (usually tertiary lecturers) had language, families, and land, and participated in cultural practices, and sometimes they organised for visits to nearby language groups. The villagers demonstrated and discussed their practices such as building houses or canoes or bows or making feasts or exchanges. We used available written records. We gathered words related to measurement. We also had reports prepared over 10 years by student teachers of cultural mathematics and its links to school mathematics. These were all analysed, and significantly visuospatial reasoning was an important aspect of their practices (Owens & Kaleva, 2008a, 2008b). Memory held many images for creating new objects, problem solving occurred regularly as a group working together on projects such as house building using ropes and sticks for assessing lengths, the eye used especially for right angles with people or sticks marking points, and talented individuals often taking the lead on specific aspects. Discussion considered purpose, angles (e.g. for slope of roof) matched to approximate heights and lengths, images of tessellated triangles and rectangles, the amount of bush material products for one area of the house, embedded mathematics such as the effect of a particular sized radius for a round house or weight for a trap, the height of the wall in comparison to the height of the person's shoulder or the floor space needed for a certain number of people, and structure and testing for balance of canoes or arrows, (see figure 5, (Owens, 2015; Owens & Kaleva, 2008a, 2008b). The student teachers linked their cultural mathematics to the school mathematics, mostly as representations in material culture but others recognised the mathematical ways of thinking of the Elders. Importantly, the students were proud that their ancestors used mathematics in their cultural activities "even if they did not know it" (as many said).

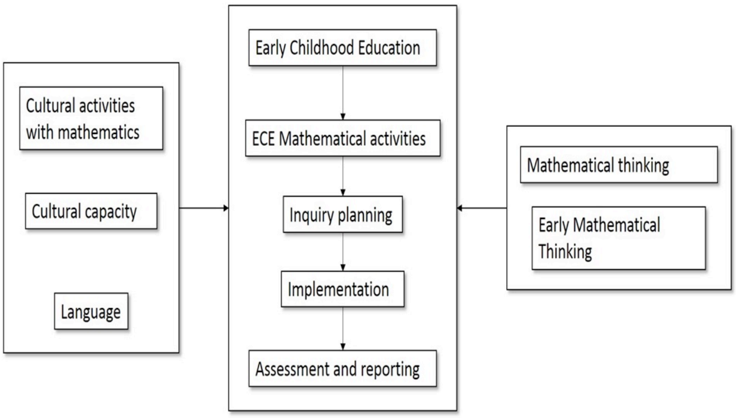
Trap model, gardening, house building (all in mid-Wahgi, Kopnung village), explaining in Miss and Malalamai, Madang, Roa Kaleva's mother (Keapara Elder) explains her basket thinking.

Figure 5. Discussing cultural mathematical activities: making a trap, garden, houses and basket.

---

[4] Rex Matang was beginning his number research building on the work on counting systems, see next section.

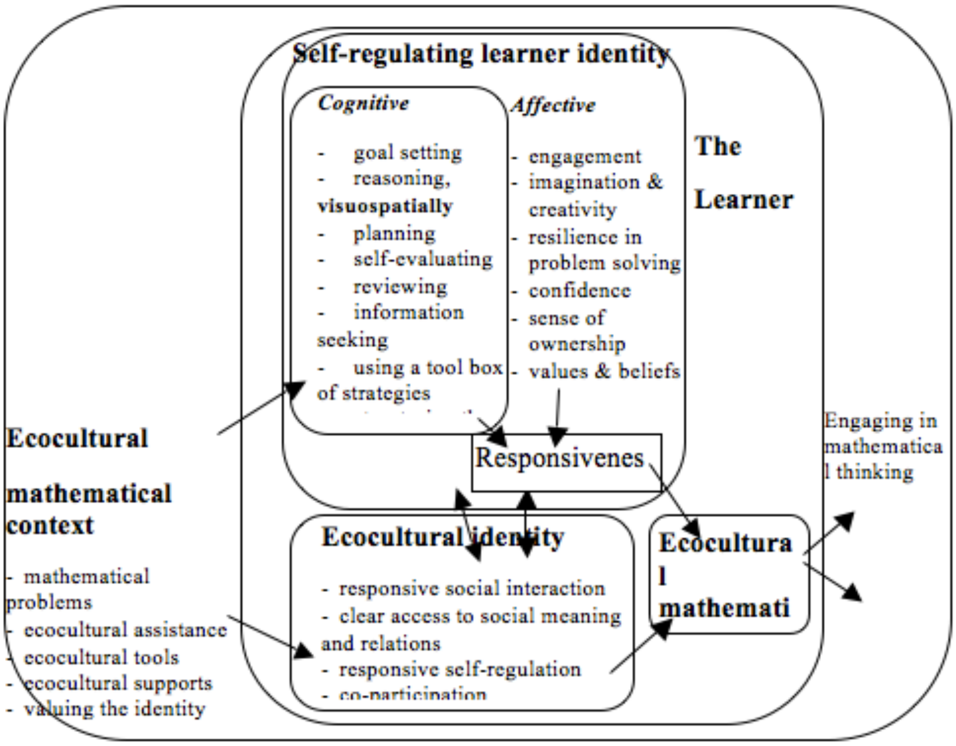
Another major design-research study was on preparing and trialing a design for PNG elementary school teachers which took account of language, cultural mathematics and early mathematics learning (at home and in the school). Lessons had an inquiry approach (Global Education Project (NSW), 2014; Murdoch, 1998). The approach has learning sections on tuning in (engaging and relating to real life), planning to find out, finding out, sorting out, going further (deeper thinking), making connections (to other mathematics, real life and other subjects), taking action (sharing with others and applying), and reflecting and summarising. Through this approach students can relate their own mathematics, and inquire about how it might be extended or related to school mathematics. Thus the connections linguistically, concretely, and then abstractly may be established in a meaningful way. During the professional learning sessions, we used videos and had teachers practice the assessment tasks and new ways of teaching. Books on mathematics topics for early readers were prepared in English with space for the local languages. We provided a manual to cover all this theory. Later, we also provided this material on computers with the associated video material, manual information, and books on mathematics topics for early readers. We provided each teacher with a solar panel, battery and computer, introducing the professional learning while teaching them how to use the computer. However, it was an aid project and the government changed and the school curricula changed but the teachers and teacher trainers learnt much about incorporating culture into school mathematics, and how to teach early mathematics effectively (Edmonds-Wathen, Owens, Bino, & Muke, 2018). The simplified final model of mathematics teaching is shown in figure 6.



(Source: Owens, Edmonds-Wathen, & Bino, 2015, p. 48)

Figure 6. Pre-elementary and elementary knowledge for teaching mathematics.

The research studies showed that reasoning visually and spatially was a critical part of mathematics and that it could be a sound way to build on cultural backgrounds. Through taking a critical ecocultural pedagogy, major change can be undertaken to enhance mathematics.



Source. Owens (2015, pp. 15, 259)
Figure 7. Culture's impact on self-regulating learner and mathematical thinking identity.

The findings from these research studies were used to develop an understanding of mathematical identity that was strongly influenced by cultural identity (Owens, 2014). Together with cultural identity, self-regulation which is dependent on visuospatial reasoning, conceptualising, attitudes and beliefs, and problem solving are keys for establishing mathematical thinking identity (Owens, 2015). This is shown in figure 7. Importantly, ecocultural contexts influence the ecocultural identity but also impact on mathematical knowledge and learning, attitudes and values. In turn this ecoculturally influenced learner identity also impacts on the mathematical thinking identity creating an ecocultural enduring ingrained valued self-perspective for ways of thinking mathematically.

3. Number Systems and Society

Building on Glen Lean's 22 year study of the counting systems of PNG and Oceania, new evidence supports his general thesis that counting did not spread in waves from the Mediterranean area but was developed by these ancient Australian and Papua New Guinea cultures. While there was some local diffusion through trade, some systems were developed within the culture, often using body parts such as hands, or all digits, or many other body parts typically up one arm, across the head, and down the other arm. Figure 8 provides one example illustrating half of the symmetrical 35-cycle body-part tally system of the Fasu in the Southern Highlands Province (May & Loeweke, 1981), one of the Trans New Guinea Phylum languages. It seems the body-part tally systems are unique to PNG and Australia.

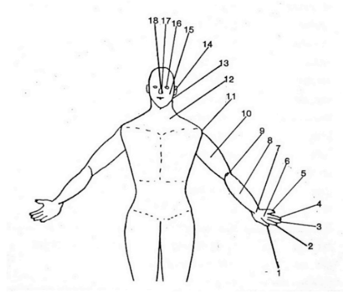


Figure 8. Body part counting system used in Namo Me (Fasu language), Southern Highlands.

Table 1. Number of languages in the collected data in each Papuan Phyla.

Types of Counting Systems	West Papua	East Papua	Torri celli	Sepik-Ramu	Trans New Guinea	Minor Phyla	Total
(2)	0	0	0	3	**39**	0	42
(2, 5)	0	1	**16**	5	86	1	**109**
(2', 5)	0	1	3	5	17	1	27
(2'', 5)	0	0	5	3	31	1	40
(5, 20)	0	1	2	**17**	52	7	**79**
(4), (4,8)	0	0	0	1	6	2	9
(6), (6, 36)	0	0	0	0	5	0	5
Body-Parts	0	0	0	8	**58**	4?	70?
(5, 10)	2	**12**	0	3	4	0	22
(5, 10, 20)	5	0	0	0	4	3	13
(10), (10, 10^{2+})	1	**8**	0	1	2	0	13
(10, 20)	2	0	0	0	1	0	3

Note. This is about two-thirds of the languages and it is limited by the use of the 1948 Hattori and Wurm atlas of languages. Several dialects are now considered separate languages and others have been identified. Bold numbers indicate a significant group. The largest phylum Trans New Guinea has a large variety with a few neighbouring ones having similar counting systems. Source. Lean (1992), emphasis added, discussed further in Owens, Lean, with Paraide, and Muke (2018).

Lean collated counting words from two-thirds of the 1200 languages recognised at the time in New Guinea and Oceania. He used government records[5], first and early contact data[6], and throughout the 1970s the questionnaire data of students and teachers and SIL people (linguists and translators) who knew the languages well (Lean, 1992). While Smith (1984) had categorised the systems of the Morobe Province into more types, Lean selected to combine the 2-cycle systems although they had higher cycles in order to assist with determining the connections between counting systems across the language groups. Table 1 shows the diversity of just the Papuan (Non-Austronesian) languages. By considering the position of languages it was noted that some of the systems spread to neighbouring older Phyla and some systems developed spontaneously but others were borrowed from neighbours probably through trade. A full account of this research is available through a book by Owens, Lean, with Paraide, et al. (2018).

Diversity is also evident among the Austronesian Oceanic languages of PNG and Island Melanesia. The clusters are given in Table 2.

[5] For example, British New Guinea and Australian administration officers were required to record details of languages.

[6] Notably the work of S. H. Ray, A. Capell, F. E. and W. G. Lawes.

Table 2. Austronesian Oceanic counting systems

Types of Counting Systems	Admiralties/St Matthias	North New Guinea	Papuan Tip	Meso-Melanesian	West Papua	Total
(2)	0	**2**	0	0	0	2
(2, 5)	0	**15**	2	1	0	18
(2", 5)	0	**7**	5	0	0	12
(5, 20)	1	**19**	11	1	2	34
(5, 10)	1	20	7	17	0	45
(5, 10, 20)	0	13	5	0	1	19
(10, 100)	23	0	9	**41**	0	73
(10, 20)	0	0	0	**3**	0	3
(4)	0	2	0	0	**2**	4
Totals	25	78	39	63	5	210

Note. Bold numbers indicate a significant group.

Examples, albeit simplified for this paper compared to the original data (cited in Owens, Lean, with Paraide, et al., 2018) are presented in Table 3. It is interesting to note that Kanum is one of a few 6-cycle system on Kolopom island (south of border between West Papua and Western Province, PNG) and on the coast nearby and further east suggesting trading links affecting the adoption of the 6-cycle system but with developments into different systems (not represented here) for larger numbers (Donahue, 2008). Cultural contexts play an important part in the development of number systems and their uses (Owens, Lean, with Paraide, et al., 2018). For example, picking up 3 taro in one hand and 3 in the other is common in the 6-cycle systems. Shell money is also associated with composite groups of 12 for counting together with collecting coconuts in groups of two then four, then six and two groups of six piling them up and counting groups of 10 so there is a word for 10 groups of 12 *a pakaruati na tangvani* (Paraide, 2018). Another example is the truncating of a body-part tally system to 20 with the 2 kina note (20 10-toea, often called 20 shillings reflecting the original introduced pounds and shillings) (Saxe, 2012) whereas for other groups the use of a 10-cycle resulted from truncating a body-part tally system for ease of representation in trade and compensation with neighbours (Franklin & Franklin, 1962). The influence of trade and of valuing those who were multilingual in their community for trade and relationship purposes (Swadling, 2010) also meant that some languages had hybrid systems, two systems, or borrowed words. In some cases, migration may have resulted in a change in language but not in all cases. Within the Austronesian group, the diaspora of a Lae group in the Markham Valley has resulted in loss of a language. Furthermore, some systems including base 10 have been replaced in Atzera in the Markham Valley by a 2-cycle counting system similar to its neighbouring Papuan languages with whom they traded (Holzknecht, 1989). Isolated pockets of languages also developed certain characteristics such as the Mumeng-Buang group.

Table 3. Examples from 8-cycle and 6-cycle systems.

Number	Hagen	Translation in g-cycle system
1	tendta/tikpa	one
2	ragl	two
3	raltika	one more than two
4	timbikak	four
5	timbikak pumb ti pip	four fingers and one thumb closing on top
8	engag/ki tendta	ki=hand=four fingers of both hands (excludes thumb)

16	ki ragl	two hands
24	ki raltika	three hands
Number	**Kanum**	**Frame words in 6-cycle system (translation)**
1	naempr	numeral (one)
2	yempoka	numeral (two)
3	ywaw	numeral (three)
6	traowao	numeral (six)
12	yempoka traowao	(two groups of six)
18	ntamnao	numeral
36	(ntaop) ptae	numeral (big)
216	tarwmpao	numeral

Note. There are also alternative phrases for these numerals. Hagen data from Lean (1992), Kanum data from Donahue (2008).

Austronesian Oceanic languages developed as base 10 systems with higher powers of 10 and some had an emphasis on pairs or the group of 10 with differing names for the group or an emphasis on completing the group. There were also some isolated variations and some influence from surrounding Papuan languages such as those that left the base 10 system for a 2-cycle system or that incorporated a (5, 20) or (2, 5) cycle system. Some places had more than one counting system. Sometimes this was a result of using suffixes or prefixes to denote the kind of object being counted. For example, in the Trobriands there are at least 42 different classes of objects or ideas including powers of 10 (numeral classifiers) that use a different suffix for each class. However, other people use different cycles for specific purposes especially certain exchanges. For example, Hagen and Gawigal have a (4, 8) cycle system but will also use a 10-cycle system especially for pig counting where one person counts to 10 and another tallies the 10s. Special systems seem to exist for specific occasions, usually restricted in attendance. For example, one Austronesian language borrowed a 6-cycle system for counting taros from their neighbour.

There are in both Austronesian and non-Austronesian (Papuan) systems, base 10 and other bases such as 4, 6, and 20, that have a systematic word pattern for large numbers; for example one of the Enga dialects, Mea, count up to 60 with each group of 4 given a name, and then that forms a new whole, Kanum has higher numerals for base 6, Iqwaye for 20. This last group is quite articulate in terms of powers of 20 with their gestural representation and links to infinity and their worldview (Mimica, 1988). Yu Wooi or mid-Wahgi marked hundreds by parts of the body (Owens, Lean, & Muke, 2018). Other people (e.g. Dobu speakers of Loboda village) understand large numbers, displaying them but not necessarily counting to large numbers (see also Paraide, 2018, on Tolai culture). A recognition of the composite group in all the counting systems from pairs (2-cycle) to twenties (5,20 cycles) and beyond (body-tally) is one significant finding. An understanding of this diversity assists the conceptualisation of the group of 10 in the base 10 systems and groups in multiplication. These systems assist the learning of arithmetic (see figure 9). Motu type languages make use of pairs for numbers 6 to 9, for example 8=2x4 and 9=2x4+1 whereas the Manus type have the number still needed to make the group, that is 6 = 10-4 or 6 and 4 more for 10.

Tok Ples	English		
2+1	3		Body Part Tally is a number line
5+1	6	Languages have a grouping (composite) unit	
5+2	7	Subtraction patterns	
2x5+1		Even and odd numbers	

Figure 9. Links between vernacular languages and English number systems.

Using evidence from archaeology, linguistics, mathematics, and oral reflections of people from continuing cultures, the longevity of counting of Indigenous cultures in Papua New Guinea and Oceania can be dated to between 40 000 and 10 000 years ago with the Austronesian Oceanic systems to 5 000 years ago. There is evidence of settlements as early as 40 000 years ago in several parts of New Guinea (see figure 10).

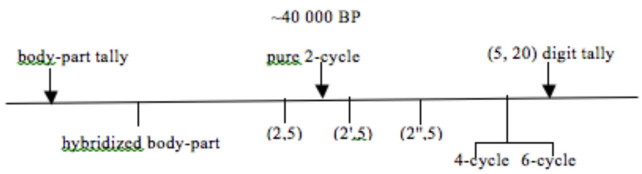


Figure 10. Papuan (Non-Austronesian) counting systems.[7]

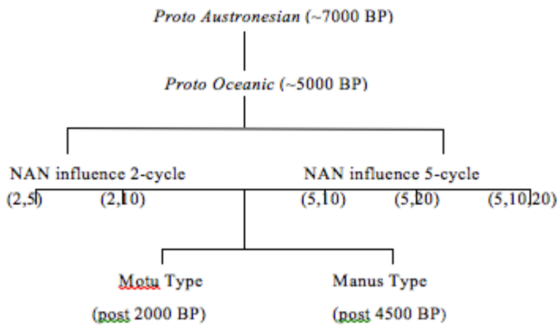


Figure 11. Genealogy of Austronesia languages.

Linguistic researchers have suggested a Proto language for the newer larger Trans New Guinea Phylum (Pawley, 2001; Ross, 2005). Ross is currently considering the number words (personal communication, 2018) for which Lean's work and more recent work on counting will be of interest. A recognition of the composite group in all the counting systems from pairs to twenties and beyond (in body-part tally systems) is one significant finding of the counting system analyses. An understanding of this diversity assists the conceptualisation of number systems such as the base 10 systems by virtue of realising that people developed alternative systems that also had a structure and pattern with different composite groups other than 10 and powers of 10. Furthermore, any history of number should recognise the mathematics of the ancient cultures of New Guinea, Melanesia, Australia, and the Pacific. The diversity of counting systems among the Papuan languages seems to relate to the identity that language affords groups. However the ocean and interrelationships that continued with migrations and trade provided the Oceanic languages a means of keeping clearer language and counting similarities.

4. Global and Local Perspectives on Curriculum

Currently I am working with Patricia paraide and Charly Muke and two other Australians to tackle the history of mathematics education in Papua New Guinea. We have recorded activities or objects that suggest foundational deep mathematical knowledge on the basis of archaeological records, practices recorded on first contact, our own cultural knowledge or experiences in villages, and the many records by students provided for the measurement study. Archeological evidence includes technological advances such as agricultural drainage systems that date back to 10 000 years ago in the highland valleys but also trade items such as pottery and obsidian and greenstone, foods and oils, ochres and plants. Trading itself and technologies require mathematical skills and many technologies and knowledges are used today such as building structures, making canoes, fishing, pottery, decorations for dance and dances, string bags and weaving, bindings and paintings.

Having established that there is in fact mathematical knowledge in the cultures of PNG, we look at the historical records of education. Interestingly over each period (ad hoc developments, colonial administrated systems, pre-Independence, after Independence, Reform, and re-Reform), the same issues arise. Who will be teaching? Who will fund? What language of instruction will be used? Who will be taught? What will be taught? and what is the purpose of education? (Owens, Clarkson, Owens, & Muke, 2019; Paraide, Owens, Clarkson, Muke, & Owens, 2020 (forthcoming)). The type of mathematics that was taught is of particular interest as well as the language of instruction and the role of vernacular languages for teaching in general and in mathematics in particular as it was often seen as outside the realm of vernacular literacy.

Mathematics education research was very strong for 15 years after Independence. However, funding was not available to continue. Much of the research focused on the PNG learner, in particular their psychological features but language and mathematical language was also considered. Recently, ethnomathematics has been an important part of the research agenda in PNG and at various times it has influenced the curriculum especially for elementary schools. For example, there was an excellent book on patterning in the first years of school during the Reform period that considered a number of PNG objects. However, 15 years later, it was hard to find. Throughout the various periods textbooks increasingly used problems relevant to PNG in terms of the contexts and words of problems. It was often left to the individual teacher to delve further into cultural practices and languages to establish links between these and the national curriculum. In the re-reform period class sizes increased even more and the teachers were faced with the new Standards in the curriculum, focusing on assessment expectations. More details for content were also included. However, English was to be the language of instruction throughout the syllabus whether or not it was spoken in the community. The impact on mathematics and ethnomathematics is yet to be felt. Increasing use of computers is also occurring with some mathematics programs in early childhood and early schooling although there use may be mostly for reinforcement rather than inquiry in mathematics.

5. Conclusion and Going Forward

This journey with Indigenous cultural research has changed my perspective and that of many others including my Indigenous and non-Indigenous co-researchers, and the large number of researchers of ethnomathematics stretched across the world. Importantly valuing Indigenous mathematical knowledges provides meaning for Indigenous students who are to live in both worlds (Burarrwanga et al., 2013; de Abreu, Bishop, & Presmeg, 2002; Paraide & Owens, 2018). The perspective values difference and for teachers to draw on the funds of knowledge of the families, communities and Elders (González, Moll, & Amanti, 2005; Lowe & Yunkaporta, 2013). However, this research enriches the mathematics of non-Indigenous students who exist in multicultural and multi-subcultural situations or those that are relatively unicultural as these students live in a global society. For example, in teaching about rectangles, I often start with the diagonals, equal ropes tied in the middle to make different floor plans keeping the ropes taut (from an African group, PNG groups check equal opposite sides and equal diagonals) and extending to what happens if the ropes (diagonals) are not equal. I have found that the properties of diagonals of rectangles are not forgotten by students who would otherwise just talk about sides and angles. Furthermore, they begin to see how rectangles, squares, parallelograms and rhombus are linked. When I teach about area and tessellations, I always use triangular and rectangular area units as well as squares to establish the meaning of an area unit. In addition, by making equilateral triangles using the method of two equal sticks used for planting cocoa trees in PNG (introduced apparently by an Indian agriculture extension officer) highlights multiple visual patterns and properties about planting for efficient nourishment of plants (in the past I had only seen plants at the vertices of squares or rectangles). Furthermore we place a bag around a new plant using the points of an equilateral triangle for the stakes. Thus mathematics is linked to agriculture and environmental studies. More sophisticated patterns such as those used in making string figures or continuous string bags (*bilums*) are often more applicable to the group who uses them as these are complex designs (some practices are not appropriate for cultural reasons). The patterns of weaving are more easily adapted to classroom use and pattern extraction and modification (Owens, Cherinda, & Jawahir, 2015).

Then there are fun problems that highlight, for example, fraction concepts. "Here is the long side of the house and you have to allocate a third for bedrooms at either end and the middle third for sitting. Here is a stick, how can you work this out without breaking the stick" (the stick is longer than a half of the length of the house but less than two-third). The story of house building and the slope of a roof (see figure 5c) is a good introduction to ratios of sides of triangles and makes meaningful the similar triangle approach to establishing tangents of angles as a ratio of sides. The emphasis is on visuospatial reasoning. In terms of number, knowing examples of (2, 5), (5, 20), 6 or 8 cycle systems that are still used today assists with teaching students about place value. Symmetry (reflective, slide, and rotational) is another area supported by PNG ethnomathematics. These are just a few examples of practical implementations of teaching mathematics. By recognising ethnomathematical practices, we value the rights of first nations' peoples and the backgrounds of immigrants and mathematical activities that have developed as a result of transcultural adaptations to the ecology in which people now live (Knijnik & Wanderer, 2015).

While there is much sharing across countries of good ways for teaching mathematics and teaching in multilingual classrooms, there may be hurdles around teacher education together with understandable curriculum that build on previous experiences but provide approaches to learning appropriate to the 21st century in a global yet local society. In that regard the inquiry approach that we utilised in the elementary school project in PNG has potential for mathematics education.

However, ethnomathematics also highlights the dangers of hegemony related to selecting overseas or outsider ways of teaching, content and language of instruction. Learning mathematical concepts from one's own background and in one's own language still needs to be accepted in many educated elite circles (Fanan, 1952; Paraide & Owens, 2018). European mathematics, relatively singular cultural countries, and those who believe in accultural mathematics continue to impact on education around the world. European reconciliation with colonised countries should recognise diversity and incorporate diversity in curricula. This achievement is not easy in postcolonial situations where neo-colonial attitudes persist and recognition of cultural knowledges and values is under-recognised by country leaders and curriculum advisers from aid organisations. Nevertheless, our elementary school project linking language, early mathematical learning and mathematical thinking provided a way of incorporating cultural knowledge into inquiry learning for the curriculum (figure 6). The PNG secondary school student teachers also saw many links between their culture and school mathematics. Ethnomathematics provides the values, knowledges, and ways of teaching that enrich teachers and students in learning mathematics.

Acknowledgements

I want to acknowledge the Elders and communities, who have shared their knowledges with us or indirectly through other researchers and for many who have made us welcome in their communities, in particular, Seronke's family in Malalamai, Kaveve, Charly's village, and Yombu's village. These projects and this paper is dedicated to you. I want to acknowledge my colleagues, too many to name, whose conversations have left their mark and for many provided enduring friendships: Glen Lean, Dan and Carrie Luke, Wilfred and Roa Kaleva and family, Yombu and Rod Selden and family, Deveni and Marion Temu, Geoff Smith, Sue and Hartmut Holzknecht, Penelope Murphy, and many others at PNG University of Technology; Rex Matang and family, Vagi Bino and family, Charly Muke and family, Susie Daino, Maclean Pikacha, Grace and Turea Wickham, Dinah Ope, Gairo and Dorcas Onagi, Michael Mel, Api and Lanis Maha, Musawe and Kay Sinebare, Mark Solon, Martin Imong, Geori Kravia, Samuel Kopamu, Priscilla Sakopa, Alan Quartermaine, and many other staff at the University of Goroka; Patricia Paraide; Serongke Sondo and family and other lecturers at Madang Teachers College; Cris Edmonds-Wathen, Philip Clarkson, Annica Andersson, Chris Owens, Andrew Pawley and Malcolm Ross, Alan Bishop, Ken Clements; Barbara Hodgkinson and SIL members; Philipa Darius, Kila Tau and others at the Curriculum Unit, PNG; Teresa Hamadi and family and others at PNG Institute of Education. Funding for projects included an Australian Development Research Award (the views expressed in this paper are not necessarily those of the Australian government), National Science Foundation (USA through PREL, Hawaii), both Western Sydney University, and Charles Sturt University for providing grants and time for projects; University of Goroka, and PNG University of Technology for accommodation and facilities together with Daniel, his family and staff at RCF Transit House Goroka.

References

Bishop, A. (1978). Spatial abilities and mathematics in Papua New Guinea. *Papua New Guinea Journal of Education. Special edition Indigenous Mathematics Project, 14*, 176-204.

Burarrwanga, L., Ganambarr, R., Merrkiwayawuy Ganambarr-Stubbs, B. G., barr, DMaymuru, D., Wright, S., Suchet-Pearson, S., & Llloyd, K. (2013). *Welcome to my country*. Sydney: Allen & Unwin.

de Abreu, G., Bishop, A., & Presmeg, N. (Eds.). (2002). *Transitions between contexts of mathematical practices*. Dordrecht, The Netherlands: Kluwer.

Donahue, M. (2008). Complexities with restricted numeral systems. *Linguistic Typology, 12*, 423-429. doi:10.1515/LITY.2008.043

Edmonds-Wathen, C., Owens, K., Bino, V., & Muke, C. (2018). Who is doing the talking? An inquiry based approach to elementary mathematics in Papua New Guinea. In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas, & D. Wagner (Eds.), *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 257-275): Sense Publishers.

Fanan, F. (1952). *Black skin, white masks*. London, UK: Paladin.

Franklin, K., & Franklin, J. (1962). The Kewa counting systems. *Journal of the Polynesian Society, 71*, 188-191.

Global Education Project (NSW). (2014). *Global perspectives: A framework for the development of active global citizenship in NSW schools*. Sydney, Australia: Author.

Goldin, G. (1987). (a) Levels of language in mathematical problem solving; (b) Cognitive representational systems for mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65, 125-145). Hillsdale, NJ: Lawrence Eribaum.

González, N., Moll, L., & Amanti, C. (Eds.). (2005). *Funds of knowledge: Theorizing practice in households, communities, and classrooms*. Mahwah, NJ: Lawrence Erlbaum.

Holzknecht, S. (1989). The Markham languages of Papua New Guinea. *Pacific Linguistics, C-115*.

Knijnik, G., & Wanderer, F. (2015). Mathematics education in Brazilian rural areas: An analysis of the Escola Ativa public policy and the Landless Movement pedagogy. *Open Review of Educational Research, 2*(1), 143-154. doi:10.1080/23265507.2015.1052009

Lean, G. (1992). *Counting systems of Papua New Guinea and Oceania.* (Unpublished PhD Thesis), PNG University of Technology, Lae, Papua New Guinea.

Lean, G., & Clements, M. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics, 12*, 267-299.

Lowe, K., & Yunkaporta, T. (2013). The inclusion of Aboriginal and Torres Strait Islander content in the Australian National Curriculum: A cultural, cognitive and socio-political evaluation. *Curriculum Perspectives, 33*(1), 1-14.

May, J., & Loeweke, E. (1981). *Fasu namo me: Fasu lexicon and grammar*. Ukurumpa, Papua New Guinea: SIL.

Mimica, J. (1988). *Intimations of infinity: The mythopoeia (cultural meanings) of the Iqwaye counting and number systems*. Oxford: Berg Publishers.

Murdoch, K. (1998). *Classroom connections: Strategies for integrated learning*. Armadale, VIC, Australia: Eleanor Curtain.

Outhred, L. (1993). *The development in young children of concepts of rectangular area measurement.* (doctoral), Macquarie University, Sydney.

Owens, K. (1994). *Solving the problem of understanding how young students solve spatial problems.* Paper presented at the Psychology of Mathematics Education Conference PME18, Lisbon.

Owens, K. (1996). Recent research and a critique of theories of early geometry learning: The case of the angle concept. *Nordisk Matematikk Didaktikk-Nordic Studies in Mathematics Education, 4*(2/3), 85-106.

Owens, K. (1999). The role of culture and mathematics in a creative design activity in Papua New Guinea. In E. Ogena & E. Golla (Eds.), *8th South-East Asia Conference on Mathematics Education: Technical papers,* (pp. 289-302). Manila, The Philipinnes: Southeast Asian Mathematical Society.

Owens, K. (2004). Improving the teaching and learning of space mathematics. In B. Clarke, D. Clarke, G. Emanuelsson, B. Johansson, D. Lambdin, F. Lester, A. Wallby, & K. Wallby (Eds.), *International perspectives on learning and teaching mathematics* (pp. 569-584). Gothenburg, Sweden: Göteborg University National Center for Mathematics Education.

Owens, K. (2014). The impact of a teacher education culture-based project on identity as a mathematics learner. *Asia-Pacific Journal of Teacher Education, 42*(2), 186-207. doi:10.1080/1359866X.2014.892568

Owens, K. (2015). *Visuospatial reasoning: An ecocultural perspective for space, geometry and measurement education*. New York: Springer.

Owens, K. (2016). Culture at the forefront of mathematics research at the University of Goroka: The Glen Lean Ethnomathematics Centre. *South Pacific Journal of Pure and Applied Mathematics, 2*(1).

Owens, K., Cherinda, M., & Jawahir, R. (2015). The importance of an ecocultural perspective for Indigenous and transcultural education. In K. Owens (Ed.), *Visuospatial reasoning: An ecocultural perspective for space, geometry and measurement education* (pp. 245-273). New York: Springer.

Owens, K., Clarkson, P., Owens, C., & Muke, C. (2019). Change and continuity in mathematics education in Papua New Guinea. In J. Mack & B. Vogel (Eds.), *Mathematics and its teaching in the Asian/Pacific region* (pp. 69-112). Hackensack, NJ: World Scientific Press.

Owens, K., & Clements, M. (1998). Representations used in spatial problem solving in the classroom. *Journal of Mathematical Behavior, 17*(2), 197-218.

Owens, K., Edmonds-Wathen, C., & Bino, V. (2015). Bringing ethnomathematics to elementary teachers in Papua New Guinea: A design-based research project. *Revista Latinoamericana de Etnomatematica, 8*(2), 32-52.

Owens, K., & Kaleva, W. (2008a). Case studies of mathematical thinking about area in Papua New Guinea. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Annual* *conference of the International Group for the Psychology of Mathematics Education (PME) and North America chapter of PME, PME32 - PMENAXXX* (Vol. 4, pp. 73-80). Morelia, Mexico: PME.

Owens, K., & Kaleva, W. (2008b). *Indigenous Papua New Guinea knowledges related to volume and mass*. Paper presented at the International Congress on Mathematics Education ICME 11, Discussion Group 11 on The Role of Ethnomathematics in Mathematics Education, Monterray, Mexico. https://researchoutput.csu.edu.au/en/publications/indigenous-papua-new-guinea-knowledges-related-to-volume-and-mass

Owens, K., Lean, G., & Muke, C. (2018). Chapter 3: 2-cycle systems including some digit-tally systems. In K. Owens, G. Lean, P. Paraide, & C. Muke (Eds.), *History of number: Evidence from Papua New Guinea and Oceania*: Springer.

Owens, K., Lean, G., with Paraide, P., & Muke, C. (2018). *The history of number: Evidence from Papua New Guinea and Oceania*. New York, NY: Springer.

Owens, K., McPhail, D., & Reddacliff, C. (2003). Facilitating the teaching of space mathematics: An evaluation. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 339-345). Hawaii: International Group for the Psychology of Mathematics Education.

Owens, K., & Outhred, L. (1996). Young children's understandings of tiling areas. *Reflections, 21*(3), 35-40.

Paraide, P. (2018). Chapter 11: Indigenous and western knowledge. In K. Owens, & Lean, G. (Ed.), *History of number: Evidence from Papua New Guinea and Oceania*. New York, NY: Springer.

Paraide, P., & Owens, K. (2018). Chapter 12. Integration of Indigenous knowledge in formal learning environments. In K. Owens, G. Lean, & P. With Paraide, & Muke, C. (Eds.), *History of Number: Evidence from Papua New Guinea and Oceania*. New York, NY: Springer.

Paraide, P., Owens, K., Clarkson, P., Muke, C., & Owens, C. (2020 (forthcoming)). *History of mathematics education in Papua New Guinea (working title)*. Cham, The Netherlands: Springer Nature.

Pawley, A. (2001). 17 The Proto New Guinea Phylum obstruents: Arguments from top-down reconstruction. In A. Pawley, M. Ross, & D. Tryon (Eds.), *The boy from Bundaberg: Studies in Melanesian linguistics in honour of Tom Dutton* (pp. 261-300). Canberra, ACT, Australia: Pacific Linguistics, Australian National University.

Presmeg, N. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics, 6*(3), 42-46.

Ross, M. (2005). Pronouns as a preliminary diagnostic for grouping Papuan languages. In A. Pawley, R. Attenborough, J. Golson, & R. Hide (Eds.), *Papuan pasts: Cultural, linguistic and biological histories of Papuan-speaking peoples* (pp. 15–65). Canberra, Australia: Pacific Linguistics.

Saunderson, A. (1973). The effect of a special training programme on spatial ability test performance. *New Guinea Psychologists, 5*, 13-23.

Saxe, G. (2012). *Cultural development of mathematical ideas: Papua New Guinea studies*. New York: Cambridge University Press.

Smith, G. (1984). *Morobe counting systems: An investigation into the numerals of the Morobe Province, Papua New Guinea.* (MPhil), Papua New Guinea University of Technology, Lae, Papua New Guinea.

Swadling, P. (2010). The impact of a dynamic environmental past on trade routes and language distributions in the lower-middle Sepik. In J. Bowden, N. Himmelmann, & M. Ross (Eds.), *A journey through Austronesian and Papuan linguistic and cultural space: Papers in honour of Andrew Pawley* (Vol. 615, pp. 141-159). Canberra, Australia: Pacific Linguistics, ANU.