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"MATHEMATIQUES ET VIVRE ENSEMBLE": POURQUOI, QUOI, COMMENT?

"MATHEMATICS AND LIVING TOGETHER": WHY, HOW, WHAT?

The indivisibles: a travel in time and space from Archimedes to Cavalieri / Les indivisibles: un voyage dans le temps et l'espace d'Archimède à Cavalieri.

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Abstract: This paper presents a teaching experiment that brings together the history of mathematics and mathematics laboratory of three-dimensional Euclidean geometry, with the use of artefacts and physical experiences. It has been realized with a class group of 24 high school students (12th grade), who were encouraged to become time traveller historians and mathematicians, investigating analogies and differences between Archimedes' and Cavalieri's methods to estimate volumes. The project had a double goal: from a research point of view, it pointed at evaluating the effectiveness of an historical inspired activity to update students' common culture about mathematics, while from a didactical point of view, the aim of this experience was exploiting the feeling of personal discovery that epitomizes hands-on activities as a pivot to promote a critical attitude towards Euclidean geometry as well as to endorse a historical approach to calculus.

Keywords: geometry, calculus, indivisibles, history of mathematics, Cavalieri

Résumé : Cet article présente une expérience pédagogique associant le laboratoire d'histoire des mathématiques et de la géométrie euclidienne tridimensionnelle à l'utilisation d'objets et d'expériences physiques. Il a été réalisé avec un groupe de classe de 24 lycéens (12e année), qui ont été encouragés à devenir des historiens et des mathématiciens voyageant dans le temps, en recherchant des analogies et des différences entre les méthodes d'Archimedes et de Cavalieri pour estimer les volumes. Le projet avait un double objectif: du point de vue de la recherche, il visait à évaluer l'efficacité d'une activité inspirée par l'histoire pour actualiser la culture commune des élèves en mathématiques, alors que d'un point de vue didactique, le but de cette expérience était de sentiment de découverte personnelle qui incarne les activités pratiques comme un pivot pour promouvoir une attitude critique à l'égard de la géométrie euclidienne ainsi que pour approuver une approche historique du calcul

Mots clés : geométrie, calculs, indivisibles, histoire des mathématiques, Cavalieri

Introduction and theoretical framework

Rethinking history of mathematics may prove to be a turning point to create meaningful classroom activities: indeed, it can be a formidable source of ideas for constructing students' mathematical skills and develop their sensitivity and interest in the evolution of mathematical concepts as well as the related symbolism and lexis.

As Marie-Anne Pech points out (Pech, 2013), mathematics consents a double journey: it makes us travel in time because our current knowledge was built by the humanity of yesterday and in space because this knowledge has to be understood in its context, taking the situated cultural approach into account.

Understanding knowledge in its context implies that "the concepts of 'mathematician' and 'scientific community' have to be differentiated according to the location and the historical period, [in fact] mathematicians are social beings and the development of mathematics is a process of interaction between mathematicians, hence, obviously there is always a social element in the history of mathematics" (Bos & Mehrtens, 1977, p.9).

In particular, we can describe European mathematicians of the XVIth and XVIIth century as pioneers who embodied the atmosphere of unearthing that characterized science in that period: "the mathematician, like an explorer, must find his way through fog and wilderness and retrieve the elusive gems. Mathematics, for them, is a science of discovery: it is about the uncovering of secret and hidden gems of knowledge. Its goals have little in common with traditional Euclidean geometry and much in common with the aims and purposes of the newly emerging experimental sciences." (Alexander, 2012, p.9).

Furthermore, Pech reminds us that, in drawing at the history of mathematics as a source for significant educational tasks, we have to consider two different aspects: the perception of history as a tool to motivate students, to humanize mathematics and to deepen the learning process, and the idea of history as an objective in itself to learn what mathematics is, to grasp its meaning, to show its constant evolutions in time and space and develop metamathematical reflections.

In this frame of thought, this teaching experiment aimed at "seeing history not only as a window from which to draw a better knowledge of the nature of mathematics but as a means of transforming the teaching of the subject itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students' development of mathematical thinking" (Furinghetti and Radford, 2008, p.626).

The inspiration came from the idea of exploiting the feeling of personal discovery that epitomizes hands-on activities to promote a critical attitude towards Euclidean geometry and to endorse a historical approach to calculus, embracing what Thomas (2015) writes in his note on Rashed Roshdi's 2011 work "D'Al-Khwarizmi à Descartes - Etudes sur l'histoire des mathématiques classiques" about "breaking the chronological boundaries inherited from political history (ancient, medieval, classical, modern mathematics), and reflect on the place of the History of Sciences, between epistemology and social sciences" (translation from the CIEAEM 70 2° announcement).

The project had a double purpose: my research interests laid in evaluating the effectiveness of a historically inspired activity to update students' common culture about mathematics. From a didactical point of view the goal of this activity was to assess if students, following a historical pattern that placed them in the position of past mathematicians, and creating with their own hands an artefact connected to a specific mathematical concept, could become more aware of the underlying mathematical meanings and be more prompted to take them in. In the long run (not included in this paper), I also aimed at laying the groundwork to observe if, getting acquainted to Cavalieri's idea that "a plane is composed of straight lines like a cloth of threads and a volume is composed of flat areas like a book of pages" (Cavalieri, 1647), could support their future understanding of the modern integration theory.

Method and activity

The activity – strictly connected to the Italian national curriculum - is conceived as a didactical transposition of Cavalieri's work, focused on the use of indivisibles to derive the formula of the volume of the sphere. The original procedure applies Cavalieri's principle (i.e. the equivalence of the volumes resulting from the equivalence of corresponding flat sections) to compare the volume of the solid delimited by a hemisphere and its circumscribed cylinder (the *bowl* in Fig.1a) to that of the cone of equal height and radius. The educational path stems from the proof given in the early 1600s by Luca Valerio - commonly known as the *bowl (scodella) of Galileo*, (Fig.1b) because Galileo reports it in his 1638 book "Discorsi e dimostrazioni matematiche intorno a due nuove scienze"– with an alternate history completion.

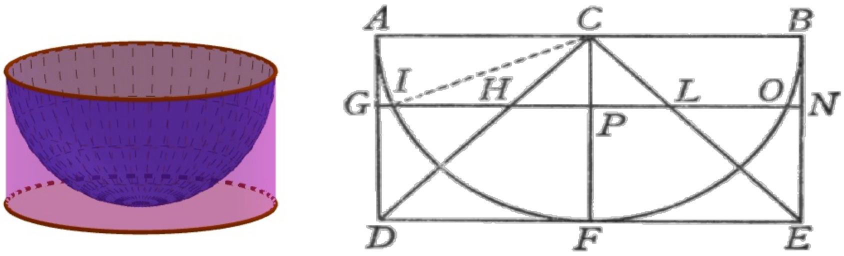

Figure 1: Galileo's bowl (a) and proof sketch (b)

Travelling back in time, the classical proof of the indivisibles equivalence is traded with its physical verification, obtained by applying Archimedes' mechanical equilibrium principle. Students are made aware that this is not the first encounter between indivisibles and levers: thanks to the rediscovery in 1906 of Archimedes' celebrated Palimpsest, containing The Method of Mechanical Theorems, we know in fact that the method of indivisibles was already used in the III century BC. Nevertheless, Archimedes did not consider it a mathematically rigorous method, therefore he used indivisibles, combined with the mechanical method, to discover the relations between areas and volumes and then he proved the same results by exhaustion.

The teaching experiment described in this paper was carried out with a class of 24 students attending the 12th grade of Liceo Scientifico (17-18 y.o.), results were gathered by the author, who was the classroom teacher, through field notes collected during the observation of the classroom activity and the following collective discussion and assessment of students' reports.

The experience is framed within the practice of the mathematics laboratory, introduced by UMI CIIM in 2001 as "not intended as opposed to a classroom, but rather as a methodology", and exploits "history of mathematics, […] as a possible and effective laboratory tool" (Bartolini Bussi, 2010 p. 42, translated by the author). In this context, students are prompted to become apprentice mathematicians and time traveller historians and are encouraged to underline and appreciate analogies and differences between Archimedes' and Cavalieri's methods to evaluate volumes.

From the procedural point of view, students are first made aware that, being independent of the postulates of Euclidean space, Cavalieri's principle is a kind of postulate itself, which provides a sufficient (but not necessary) condition for the equivalence of two geometric figures. On the other hand, Cavalieri's principle becomes also necessary when the two solids have equal heights, therefore, if we derive the equivalence of the bowl and cone from a different proof, we can deduct the equivalence of the indivisibles and verify it with the law of the lever. Thus, they are initially guided in applying sufficient Cavalieri's principle in the classical proof of the equivalence (i.e. the equality in volumes) between the sphere and its circumscribed hollowed-out cylinder (Fig.2).

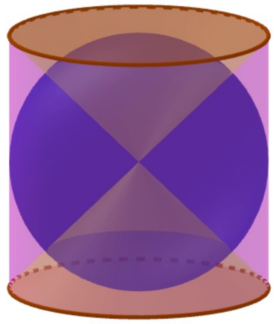

Figure 2: the setting for the classical proof

Students are then divided into eight groups and assigned the following task:

- *prove* the equivalence of the cone and bowl deriving it from the classical proof and connect it to that of the corresponding flat sections;
- assuming that the common height of the solids is 7.5 cm, *calculate* the exact measurements of the assigned flat sections (see Fig. 3) and cut them out using the provided foam sheets (Fig. 4);
- *verify* the equivalence of the flat sections using Archimedes' procedure: equality of mechanical moments (Fig. 4) and obtain the equivalence of the indivisibles from this result.

In this frame of work, the proof part comes before the verification in order to draw students' attention to the conceptual difference between the two steps.

To assign a specific level to each group, a dynamic figure created using GeoGebra was shared with the class: students could drag the horizontal line to their assigned level and refer to the ruler to read the measure indicating the position of their indivisibles (Fig.3).

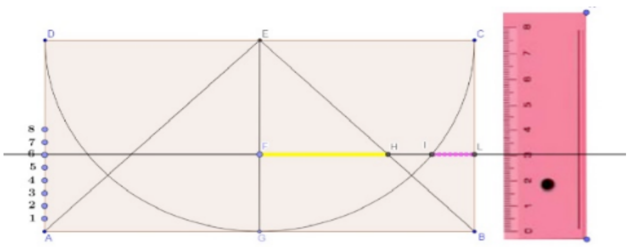

Figure 3: the assignment of the sections

Working in groups, students had then to figure out the math part needed to appraise the measurements of their indivisibles and cut them out from the provided foam sheets. Finally, they constructed the lever arms using pierced wooden sticks and checked for the expected equilibrium (Fig.4).

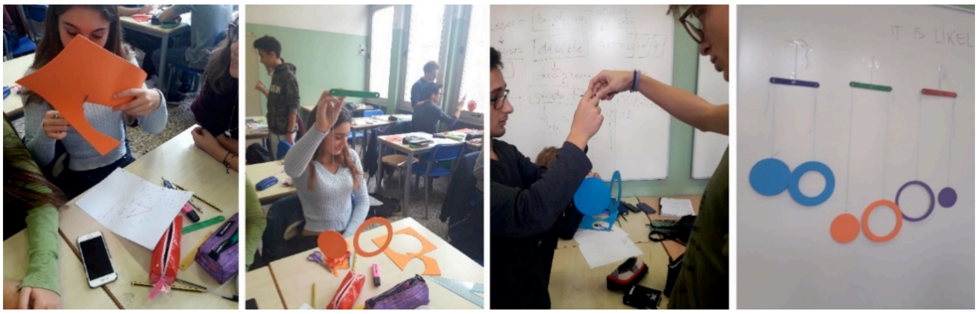

Figure 4: the indivisibles construction and the equilibrium checking

In conclusion, class discussion allowed the sharing of results and findings and then each group summarized his conclusions in a written essay at home, reports were submitted and graded by the teacher.

Comparing aims to students productions

Excerpts from field notes, students’ lab reports (indented quotations) and reflections emerged during class discussion, made it possible to assess whether Furinghetti and Radford’s cited goal of “enhancing the students’ development of mathematical thinking” had been met or at least approached.

1 - Cavalieri’s principle as a necessary condition:

> “Since the bowl and the cone have the same volume and the same height, if we cut off both solids with a plane parallel to the common base, for the Cavalieri's principle [the sections] will have equivalent surfaces.”

This excerpt supports the impression that students are aware of the additional hypothesis needed to apply the Cavalieri principle as a necessary condition.

2 - The Maths behind it: the calculations of the measures of the sections

During this evaluation part, weaker students had a hard time calculating the exact measures of their indivisibles, but they were strongly motivated to succeed in order to finalise their construction, while a group of stronger ones underestimated the problem and cut wrong sections, but were confronted with their mistake once the lever arm was not balanced. These episodes sustain the effectiveness of the laboratory approach.

3 - The Physics behind it: verification vs proof

> “With Archimedes’ procedure, we verify in a physical way that the section of the cone (circle) is equivalent to that of the bowl (annulus) […] To carry out Archimedes’ procedure we need material sections.”

The accurate use of the term “verify” enforces the idea that students correctly grasped the difference in question and that, although Cavalieri’s indivisibles are one dimension less than the continuum they generate, we need to give them some thickness in order to make Cavalieri and Archimedes meet.

4 - Mathematical equivalence vs Numerical coincidence: comparing surfaces

> “The two results have a minimal difference due to the approximations made during the carrying out the calculations”

This shows that students are aware of the necessity of allowing for errors and of the difference between irrational numbers and their rational approximations.

5 - Indivisibles or infinitesimals? The birth of modern calculus

During the discussion, the utterance of a student: "an indivisible is a plane figure of infinitesimal thickness" eased the shift from historical to epistemic awareness. In fact, it triggered a metamathematical (in Pech's sense) class talk about the difference between indivisibles and infinitesimals and about the path that, from Cavalieri to Newton and Leibniz, allows the morphing from indivisibles to infinitesimals, that leads to the birth of modern Calculus. In the discussion students' attention was drawn to the fact that Cavalieri knew that this method of summing lines into areas and areas into volumes could hide some pitfalls, but that, embodying the experimental thrust of that historical period, "he was less interested in questions as to the precise nature or existence of indivisibles, than in their pragmatic use as a device for obtaining computational results. Rigour, he wrote in the Exercitationes, is the affair of philosophy rather than mathematics" (Edwards, 1979, p.104).

Results and conclusions

Observing the processes, the outcomes of this activity seem to support the efficacy of a history-inspired laboratory strategy to engage students and promote their mathematical activity. The final verification through the physical perception of the equilibrium deeply involved them, they rejoiced visibly when the lever arms stood still in the equilibrium position and they wondered whether Archimedes had felt the same way. Archimedes, Galileo and Cavalieri emerged from the past and from the stillness of the textbook to act as workmates, engaging students, humanizing the learning process, and fostering the students' awareness of the relevance of the history of mathematics and its role within our culture.

Several different levels of mathematical bearings were synergically integrated with the historical aspect, that students eventually had to convey in their final papers: a geometric one to understand the cross-section image of the bowl and grasp the 3D shape of the sections, a symbolic one to derive a formula to obtain the sizes of the required sections and a numerical one to finally compute the exact measurements. Artefacts, tools and sign systems acted as effective means for the construction of knowledge: these rich stimuli, together with the act of constructing the mathematical objects themselves, activated a range of intertwined referents which effectively supported the students' learning, prompting them to review the fundamental difference between verification and proof and in the same time hopefully smoothing the path for the future introduction of Calculus.

References

Alexander, A., (2012) From Voyagers to Martyrs: Toward a Storied History of Mathematics, in *Circles Disturbed*, A. Doxiadis, B. Mazur (Eds), Princeton University Press

Bartolini Bussi, M.G., (2010). Laboratori delle macchine matematiche per l'Emilia Romagna, in *Scienze e Tecnologie in Emilia-Romagna*, Martignone F. (Ed.) Napoli: Tecnodid

Bos, H. and Mehrtens, H., (1977) *The Interactions of Mathematics and Society in History. Some Exploratory Remarks.* HM 4,7-30

Cavalieri, B., (1647) *Exercitationes Geometricae sex,* Monti, Bologna

Edwards, C.H., (1979) *The Historical Development of the Calculus*, Springer-Verlag

Furinghetti, F. & Radford, L., (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In *Handbook of International Research in Mathematics Education*, L. English (Ed.), New York: Routledge, Taylor and Francis.

Galilei, G. (1638) *Discorsi e dimostrazioni matematiche intorno a due nuove scienze,* volume VIII, Leiden

Pech, M. A., (2013) *Utilisation des textes historiques en mathematiques a l'ecole elementaire*. Education.

Thomas, E., (2015) *Notes de lecture*. Tangente, 167, Nov., http://tangente-mag.com/note_de_lecture.php?id=109 retrieved Sept 2018

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Promoting Language Awareness and Integrating Intercultural Learning into Mathematics Teacher Education: Concept of a Joint Seminar for Teacher Students and Refugee Teachers

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Abstract: Increasing diversity in linguistic and cultural backgrounds of German school students requires teachers to integrate language and intercultural learning into their teaching. Contrary to popular belief, students' language skills play a major role in learning mathematics as they affect the ability to communicate as well as comprehend mathematical ideas. Teacher training often focuses on theoretical knowledge and teacher students don't feel qualified enough to put their knowledge into practice. In order to tackle this problem, a multi-stage seminar was designed that gives math teacher students the chance to apply integrated language learning in practice by designing and conducting a seminar for refugee teachers. Aspiring to teach mathematics in German schools, the refugee teachers benefit from participating in this seminar by improving language skills and vocabulary regarding school mathematics. Through their encounter, both groups of participants are given the opportunity to gain intercultural experiences, which prepares and equips them for their future teaching.

Résumé: La diversité croissante des origines linguistiques et culturelles des élèves allemands oblige les enseignants à intégrer l'apprentissage linguistique et interculturel dans leur enseignement. Contrairement aux idées reçues, les compétences linguistiques des élèves jouent un rôle majeur dans l'apprentissage des mathématiques, car elles affectent la capacité de communiquer et de comprendre les idées mathématiques. La formation des enseignants est souvent axée sur les connaissances théoriques et les étudiants ne se sentent pas suffisamment qualifiés pour mettre leurs connaissances en pratique. Afin de s'attaquer à ce problème, un séminaire en plusieurs étapes a été conçu pour donner aux étudiants en mathématiques la possibilité de mettre en pratique l'apprentissage intégré des langues en concevant et en organisant un séminaire pour les enseignants réfugiés. Aspirant à enseigner les mathématiques dans les écoles allemandes, les enseignants réfugiés bénéficient de la participation à ce séminaire en améliorant les compétences linguistiques et le vocabulaire relatif aux mathématiques à l'école. Grâce à leur rencontre, les deux groupes de participants ont la possibilité d'acquérir des expériences interculturelles, ce qui les prépare et les prépare pour leur futur enseignement.

Language and Mathematics Teaching

German school students are as diverse as they have never been before. Due to various reasons of political or societal nature, pupils from different cultural, social and linguistic backgrounds have the opportunity to learn together and from each other. Heterogeneity of students quite naturally entails heterogeneous language skills, a fact that concerns also teachers of mathematics. It has long been claimed that mathematics is a language in itself which is universally spoken and thus overcomes language barriers. Whilst being a nice metaphor that appropriately highlights certain aspects of mathematics, it runs the risk of playing down the very complex role of language in mathematics learning (Barwell, 2005). Language evolution itself occurs in two steps: an auto-organizational advancement of concepts from core cognition and their following externalization through communication in social interaction (Reboul, 2017). Both the cognitive and communicative function of language (Maier & Schweiger, 1999) remain of lifelong importance and can be identified in mathematics learning whenever language is used as either an internal tool to comprehend mathematical concepts and ideas or as a tool of externalized communication and presentation of the latter (Prediger & Zindel, 2017). In other words, we need language to both think and talk about mathematics. Concerning the practice of mathematics teaching in heterogeneous classrooms, language sensitivity can be regarded as part of a broader range of intercultural competences prospective teachers need to acquire in order to meet their future students' diversity (Bishop et al., 2015).

Language Awareness in University Math Teacher Training

In order to foster students' participation in communication both inside and outside of the classroom, teachers should have an awareness of the impact of language on mathematics learning as well as the motivation and qualification to integrate language and subject learning in their teaching. Encouraging either is hence a crucial task of university teacher training and therefore, the chair of mathematics education at the University of Potsdam, Germany, offered its master students a seminar on language sensitivity in mathematics teaching. In this seminar, teacher students met on a weekly basis to learn about common linguistic theories as well as ways and means of supporting language learning in mathematics. The course curriculum contained among other things self-experiments on learning mathematics in a foreign language, learning about the importance of employing different representational registers of a mathematical content and detecting stumbling blocks of the German language, which often depict an obstacle for learners with lower language skills but tend to be invisible and overlooked by native German-speaking teachers. The seminar concluded in students planning a teaching unit specifically focusing on integrating both conceptual and language learning, which was ultimately presented to fellow students but unfortunately not tested out in real classrooms.

An overall idea of the seminar has been to offer teacher students a comprehensive experience of integrated language learning. Accordingly, the recurring character of the seminar sessions was a praxis orientation as close as possible by working with video material, school books and, last but not least, integrating the lecturer's experience and material collected whilst teaching at the trilingual German School Montevideo in Uruguay. The students' evaluation of the seminar has been very positive and proven right the presumption that the majority of them does not come across any comparable theoretical content input during their three to four years of teacher training but that they are highly motivated to integrate language learning in their future mathematics teaching. Furthermore, none of them seems to have a chance to gather practical experience of integrated language learning, which unfortunately the seminar did not provide them with either. Approaching this issue and trying to pursue the initial idea of a more comprehensive seminar experience, the chair of mathematics education reached out for opportunities for its teacher students to gain practical experiences and this is how a collaboration with the *Refugee Teacher Program* was initiated.

Refugee Teacher Program

The Refugee Teacher Program (RTP) is a qualification program at the University of Potsdam that was established in 2016 and is addressed to trained and experienced teachers who had to leave their home countries because of war or persecution. Many of the refugee teachers have years of teaching experience and continued teaching even during their flight, but now they are faced with the impossibility of working as teachers in German schools due to their lack of language skills or difficulties concerning the recognition of educational qualifications. The RTP aims at integrating these teachers into German schools by providing them with German language skills and giving them an insight into the German school system. Employing refugee teachers is not only a logical way to react to the increasing lack of teachers in German schools in general, but makes particularly sense with regard to the teaching of thousands of school students who arrived as refugees to Germany as well and the communication with their parents. (Vock, 2017)

The RTP is a three-semester program that comprises a full-time German course during the first semester, practical experience and classes on pedagogy and teaching methodology during the second semester, and another intensive German course during the third semester followed by final exams. Participants are expected to reach a language level of C1, which corresponds to the general level of language proficiency required for entering any program at a German university. Each semester there are around 30 to 40 refugee teachers commencing the program. All teachers who finished the program successfully have been granted job contracts at German schools (Vock, 2017).

Much of the refugee teachers' time is spent in language courses or at schools collecting practical experiences and, so far, cooperative encounters with regular teacher students from the University of Potsdam seem to be relatively rare. This is quite lamentable for several reasons. Firstly, refugee teachers could benefit from the knowledge of those who experienced the German school system themselves and can thus provide first-hand information about curricula, grading, etc. Second and most importantly, German teacher students can support refugee teachers with obtaining certain language skills that are necessary for teaching in German schools. These skills do not only include knowing subject-specific German vocabulary, which quite honestly would be challenging enough. Teachers are also required to have a deep understanding of the three different verbal registers, the *subject-specific, general academic* and *daily language* (Smit et al, 2018), that are all equally used in classrooms and between which they need to be able to translate fluently as this is what is ultimately expected from their students as well.

Concept of a Joint Seminar

The chair of mathematics education at the University of Potsdam saw an opportunity to tackle the above-mentioned problems by giving its previous seminar on language sensitive mathematics teaching a re-design and collaborating with the RTP. The result was a multi-stage seminar concept that on the one hand provides German teacher students with the opportunity to gain practical experience in integrated language learning and on the other hand supports refugee teachers in their learning about language of school mathematics.

Structure and Organization

The newly designed seminar comprises three consecutive parts: (1) a two-day seminar for mathematics teacher students during the first week of the semester, in which they acquire the same theoretical knowledge as in the previous weekly seminar on language sensitivity in mathematics teaching, (2) an ensuing one-day workshop for the same students to design a seminar for refugee teachers on language and vocabulary in school mathematics, which is realized in (3) a weekly seminar on language in school mathematics held *by* German mathematics teacher students *for* refugee mathematics teachers. Phase (3) is accompanied by interim and final evaluation sessions, in which the German teacher students have the opportunity to reflect on their previously held seminar sessions and to discuss possible ways of improvement before holding another seminar session. This way, they go through a continuous loop of analyzing, implementing and reflecting their own seminar sessions. Depicting a common method presented in further teacher training to guarantee and improve teaching quality (Primas, 2013), it seems more than sensible to apply this loop in university teacher training already. To activate the German teacher students' participation in the seminar even further, they were also asked to reflect on their seminar concept itself, that they designed in phase (2) and conducted in (3), ongoingly and to alter it whenever necessary.

Aims

The joint seminar is expected to have positive outcomes for both participating groups, the German teacher students as well as the refugee teachers. For one thing, the seminar aims at giving mathematics teacher students the opportunity to experientially comprehend the importance of language in mathematics teaching. They are believed to do so by getting in contact with fully trained and experienced mathematics teachers who are not yet allowed to teach in German schools, in fact, not due to their lack of mathematical knowledge but solely due to their lower language skills. This extreme contrast is believed to emphasize the significance of language in mathematics teaching quite forcefully. Furthermore, teacher students are given the opportunity to put into practice their seminar learnings about language sensitive mathematics teaching and to play a creative role in the development of a seminar itself. While designing and conducting their own seminar for refugee teachers, the teacher students have the chance to apply their theoretical knowledge and hence enrich it with practical experience. On the side of the refugee teachers, the seminar aims at broadening language skills to subsequently enable them to communicate about school mathematics in the different verbal registers of the German language. This will hopefully help the refugees to feel well-equipped for their future teaching of mathematics and thus most likely raise their motivation for working in German schools.

The collaboration between German teacher students and refugee teachers during the seminar aims at creating an environment for intercultural learning between participants. Given the variety of cultural backgrounds among school students, intercultural learning is an important task of school. In order to foster intercultural competence among students it is necessary that teachers possess or develop this competence themselves (Busse & Göbel, 2017). Whilst part of this is one's personal responsibility, university teacher training can definitely contribute to it by facilitating intercultural encounters. For tangibility reasons, three different levels of targets are mapped out on which intercultural learning could take place during the joint seminar. On an affective level, seminar participants are expected to open up towards different cultural backgrounds of people and develop a basic attitude of curiosity and fondness rather than aversion towards each other. Hopefully, they will also learn to see their own or other people's multilingualism as a competence that is worth to be deployed and developed in mathematics teaching. On a cognitive level, refugee teachers and teacher students will presumably come across similarities and differences in conducting mathematics and teacher training between Germany and other countries and thus explore the relationship between mathematics, culture and anthropology, the so-called *ethnomathematics* (D'Ambrosio, 1985). Last but not least, on a level of action, both participant groups will be enabled to communicate with each other on general and more specific topics such as school and school mathematics. Additionally, teacher students, who are in charge of conducting the seminar, will develop the ability to integrate cultural differences into their seminar session quite spontaneously. It goes to show how manifold and in what detail intercultural learning can occur during the collaboration.

First Results

The first run of the above-described joint seminar took place in the summer semester 2018 with eight German teacher students and five refugee teachers from Syria. Since the weekly seminar was designed for non-native German speakers with a connection to mathematics, an Italian Erasmus student and a French intern from the mathematics department could participate as well. The entire joint seminar was guided and accompanied by one of the authors of this paper in her function as member of staff of the chair of mathematics education and lecturer of the seminar on language sensitive mathematics teaching.

With the objective to evaluate on the concept and develop it further, the joint seminar was accompanied by a small explorative research that aimed at giving an overview of participants' impression of the seminar and a first insight into the seminar's impact on its participants. At two (refugee teachers) or three (German teacher students) times throughout the semester, participants were asked to fill out a questionnaire in order to see if a development towards the aims above had taken place. Parts of the oral discussions during the interim and final evaluation sessions were also taken into consideration for these first results.

The questionnaire revealed that the *importance of language in mathematics teaching* became obvious to German teacher students in many ways. Some reported that their difficulty to grasp refugee teachers' ideas, even though it was about a familiar mathematical topic, ultimately led to the use of gestures, drawings and written formulas. Others found themselves unable in a situation of explaining frequently used words from technical, educational and everyday language because their use was so automatic and never been thought about before. One teacher student wrote that "it felt peculiar to talk to educated adults in a way that one would usually talk to children", which very forcefully shows how much mathematics teaching depends on language skills. For most teacher students the opportunity to design and conduct a seminar (session) on their own depicted an appreciated way to acquire both *theoretical knowledge and practical experience.* Regarding their creative role in the course, a participant noted that she had "never before attended a course like this where participants could also collaborate in the realization of a course". Several of the German teacher students showed interest in taking part in next semester's course as well, this time voluntarily and without gaining credit points. Refugee teachers said that the course helped them not only *broadening their vocabulary* but "seeing different school realities" and learning more about the German education system. All of the five participating refugee teachers asked to continue taking the course during the next semester and to even meet twice a week, because they felt it was one of the most useful seminars for them in terms of feeling *prepared and motivated to teach* in German schools.

With regard to the intercultural learning targets, many positive outcomes in both participant groups could be seen. German teacher students showed a very positive *basic attitude* towards different cultural backgrounds and their inclusion into teaching. Many said that different cultures should be embraced as they enrich the mathematics class room as well as broaden the teacher's horizon. *Multilingualism* was described as something very valuable, as it "can throw a different light onto certain mathematical content and thus offers new access to it" (German teacher student). On the same topic, a refugee teacher revealed that "mathematics is even more interesting in other languages". A recurring topic in the weekly seminar sessions was how mathematics is done and taught differently in the participants' home countries. A lecturer assistant was in charge of documenting all those *cultural differences in conducting mathematics* that came up throughout the seminar. Over the time span of one semester already, this built up to a collection that can be of interest to any prospective mathematics teacher and will be further used in the university's teacher training. Last but not least, an overall increase in both *motivation and qualification* to integrate language learning into mathematics teaching could be recorded among German teacher students.

Outlook

Due to its quite successful first run, the joint seminar will be offered again in the following semester. This way, refugee teachers' and teacher students' wishes to participate again can be fulfilled and further national students and interested international people can become a part of the collaboration between the chair of mathematics education and the RTP. After having captured a very general representation of seminar participants' opinion on the seminar during the first run, a more detailed look into how the seminar's objectives on the above mentioned levels of affection, cognition and action should be generated through the second one. In the future, it could be also interesting to expand the idea of a collaboration between teacher students and refugee teachers also to other departments or faculties of the University of Potsdam or even other national and international universities. The most important reason for spreading the idea is that higher education needs to become aware of and fulfill its role in integrating refugees, which it very simply can by providing them with participation opportunities.

References

Barwell, R. (2005). Language in the Mathematics Classroom. *Language and Education*, 19(2), 96-101.

Bishop, A., Tan, H., & Barkatsas, T.N. (2015). *Diversity in mathematics education: Towards Inclusive Practices*. Cham: Springer.

Busse, V. & Göbel, K. (2017). Interkulturelle Kompetenz in der Lehrerinnen- und Lehrerausbildung: Zum Stellenwert unterkultureller Einstellungen als Grundlage relevanter Handlungskompetenzen. *Beiträge zur Lehrerinnen- und Lehrerbildung*, 35(5), 427-439.

D'Ambrosio, Ubiratan (1985). Ethnomathematics and Its Place in the History and Pedagogy of Mathematics. *For the Learning of Mathematics,* 5(1), 44-48.

Maier, H. & Schweiger, F. (1999). *Mathematik und Sprache. Zum Verstehen und Verwenden von Fachsprache im Mathematikunterricht.* Vienna: Öbv & Hpt.

Primas (2013). *Guide for professional development providers.* Retrieved from http://primas-project.eu/wp-content/uploads/sites/323/2017/11/FINAL_WP4_Guide_PD_providers_licence_150708.pdf (27.09.2018)

Prediger, S. & Zindel, C. (2017). School academic language demands for understanding functional relationships - A design research project on the role of language in reading and learning. *Eurasia Journal of Mathematics Science and Technology Education*, 13(7b), 4157-4188.

Reboul, A. (2017). *Cognition and communication in the evolution of language (Oxford Studies in Biolinguistics)*. Oxford: Oxford University Press.

Smit, J., Gijsel, M., Hotze, A. & Bakker, A. (2018). Scaffolding primary teachers in designing and enacting language-oriented science lessons: Is handing over to independence a fata morgana? *Learning, Culture and Social Interaction,* 18, 72-85.

Vock, M. (2017). Refugee Teachers Program. Ein Qualifizierungsprogramm für geflüchtete Lehrerinnen und Lehrer an der Universität Potsdam. *SchulVerwaltung. Fachzeitschrift für Schulentwicklung und Schulmanagement*, 28(19), 274-276.

Translanguaging in Malta: Teaching mathematical concepts in Maltese and English

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Abstract: This paper focuses on the use of two languages for teaching mathematical concepts in multilingual classrooms in Malta. The main aim of the present study is to investigate when teachers use Maltese or English for which purpose when introducing mathematical concepts. The interlacing of Maltese and English in the context of translanguaging in teaching is illustrated by a case study methodology, whereby this paper focuses on two cases to illustrate how a fourth-grade teacher introduces the concept of weight and how a sixth-grade teacher introduces percentages. First results show that both languages fulfill different functions during mathematical instruction, all of which are necessary for efficient mathematical teaching and learning under consideration of the teacher' and students' multilingual background.

Résumé : Cet article porte sur l'utilisation de deux langues pour l'enseignement de concepts mathématiques dans des classes multilingues à Malte. L'objectif principal de la présente étude est d'étudier les cas dans lesquels les enseignants utilisent le maltais ou l'anglais dans quel but pour introduire des concepts mathématiques. L'entrelacement du maltais et de l'anglais dans le contexte de la traduction de l'enseignement est illustré par une méthodologie d'étude de cas. Cet article se concentre sur deux cas illustrant comment un enseignant de quatrième année introduit le concept de poids et comment un enseignant de sixième année introduit des pourcentages. . Les premiers résultats montrent que les deux langues remplissent des fonctions différentes au cours de l'enseignement des mathématiques, nécessaires pour un enseignement et un apprentissage efficaces des mathématiques, en tenant compte des antécédents multilingues de l'enseignant et des étudiants.

Aims of study

During times when European classrooms are becoming increasingly multilingual, teachers are faced with new challenges concerning sustainable teaching of mathematical concepts whereby language(s) is an important teaching and learning medium. Teaching multilingual students whose first language differs than the language of education requires teachers not only to include language learning in content learning, but also to make sure that the language of instruction does not impede student learning of mathematical content. In this present study, I investigate how multilingual primary teachers take into account language diversity in mathematics classroom and how they use translanguaging as a tool for teaching mathematical content in order to enrich students' mathematical learning. Hence the main research question of this present study is: How do multilingual primary teachers use two languages for teaching mathematical content?

Theoretical Framework

Spagnolo & Di Paola (2010) state that an effective teaching of mathematics requires teachers to take the socio-cultural background of students and other culture-related phenomena which influence the way how students think into account. For instance, this means that the child's home language should be involved in the learning processes if the general conditions are fulfilled (e.g., teacher speaks the concerning languages). A phenomenon which reflects this flexibility among language use during learning and teaching processes is translanguaging. Translanguaging denotes the "act performed by bilinguals of accessing different linguistic features or various models of what are described as autonomous languages, in order to maximize communicative potential" (Garcia, 2009, p. 140). Similarly, Baker (2011, p. 288) describes translanguaging as the "process of making meaning, shaping experiences, understandings and knowledge through the use of two languages". Hence, translanguaging does not merely denote the use of two languages in utterances or discourse, i.e. code-switching, but focuses more on how code-switching influences learning and teaching processes. An analysis of the use of two languages for teaching mathematics in multilingual classrooms requires an account of socio-cultural norms which might be initiated by a particular language and the understanding to which extent the language proficiency of both languages might be different. For instance, the context and topic in discourse tends to play an important role in which language is more dominant in the underlying communicative processes. Other important aspects are whether the languages have been firstly or secondly acquired, whether the everyday, educational or technical register of the particular language is viable, and the role of the languages in the educational system.

Research Methods

Socio-cultural Background

Malta is a small nation island in Southern Europe, between North Africa and Italy. The native language is Maltese, a language originating from Siculo-Arabic and is strongly influenced by borrowing from Italian and English. However, as a former British colony, English, another official language, is also important in the education system, especially in mathematics and science classrooms, is the goal of learning, as the following excerpt from the Maltese primary syllabus for mathematics indicates:

> As they grow children should be encouraged to express and articulate their explanations, thinking and reasoning in English to strengthen their mathematical communication skills. However, on no account should the use of either language (Maltese or English) impede upon the children's learning of mathematics (Ministry of Education and Employment [MEE], 2014, p. 10)

Nevertheless, teachers are expected to show flexibility of language use and "decide what language must be used to facilitate the development and acquisition of mathematical concepts" (MEE, p. 10).

Methods of Data Collection and Analysis

In order to observe teachers' use of Maltese and English in primary classrooms, classroom observations together with audio recordings of lessons will be conducted in a case study methodology. Due to the fact that the focus is on the teacher's use of languages during the acquisition of mathematical concepts, it is believed that cases provide better understanding of this phenomenon. The long-term goal is that several different teachers participate in this study, each case contributing to a development of a case study, since teacher are highly responsible for the use of both languages in her or his mathematics classroom. The teacher lessons to be observed and audio recorded were chosen in a way that the teacher introduces a new topic of a unit. This present study focuses on a teacher teaching a fourth-grade and another teacher of a sixth-grade classroom in a primary school on the island of Gozo in Malta, where most teachers and primary students have Maltese as their first language and English as a second language. Two units in the fourth grade consisting of two lessons were observed and recorded, a unit about introducing the concepts of weight and mass and a unit about introducing the concept of fractions and their equivalency. In the sixth grade, a lesson about the introduction of percentages was observed and recorded. The recorded lessons were transcribed and analyzed regarding the use of languages for teaching the mathematical concept(s) using a qualitative interpretative approach.

Results

As indicated in the aim of this study, the main focus is the analysis of the teacher's language use during the introduction of a mathematical concept. The following transcript excerpt illustrates the use of two languages in discourse between the fourth-grade teacher (T1) and students (S) in the first phase of the introduction of weights in circle time in the class:

Turn		Original	English translation
1	T1	Qabel il-half yearly semmejna is-similes. Għadkom tiftakru x'inhuma is-similes?	Before the half yearly we named similes. Do you remember what they are, similes?
2	S	(in choir) Iwaaa.	(in choir) Yesss.
3	T1	X'inhuma?	What are they?
4	S1	As cold as ice.	As cold as ice.
5	T1	(confirming) As cold as ice. (...) Din x'inhi (showing a feather)?	(confirming) As cold as ice. (...) What is it? (showing a feather)?
6	S3	As light as a feather.	As light as a feather.
7	T1	Araw hux light veru (gives the feather to the students)! Xi tfisser light (to Student 4)?	Look whether it is really light (gives the feather to the students)! What does light mean (to Student 4)?
8	S4	Ħafifa.	Light.
9	T	Ara hux ħafifa! Ħafifa?	Look if it's light. Is it light?
10	S4	Iwa.	Yes.
11	T1	Ejja, pass it on. Araw hux ħafifa. (...) U semmejna simile oħra. U x'ħin semmejna din il-kelma kien hemm min qalli: Dak x'ikun?	Come on, pass it on. Take a look whether it's light. (...) And we named another simile. And when we named it, this word, someone told me: What is it?
12	S5	Xi ħaġa ċomb.	Something [to do with] lead.
13	T	U bl-ingliż jgħidulha?	And in English what is it called?
14	S5	Ċomba.	[one piece of] Lead.
15	S3	Lead.	Lead.

16	T1	Bl-ingliż jgħidulha lead. Din (showing a piece of lead) jużawha... din ġibta mingħand id-daddy tiegħi... jużaha biex jistad... biex il-ħabel l-irqieq, dak li jkun ħafif, idendilha miegħu biex il-ħarira tinżel l'isfel, fejn ikun hemm il-ħut. Araw kemm hi tqila... Mhux bħal-feather. U x'inhi dik is-simile?	In English, it is called lead. This (showing a piece of lead) is used... I brought this from my dad... he uses it for fishing... so that the thin rope, which is light, he attaches it [the lead] to it [rope] for the thread to go down, where the fish are. Look how heavy it is... Not like the feather. And what is that simile?
17	S4	As heavy as lead.	As heavy as lead.

Table 1: A transcript excerpt from the first phase of introducing the concept of weight in Grade 4

In the above episode, the teacher starts the unit of weight by focusing on recalling the concepts of lightness and heaviness, which are important for understanding the concept of weight. She achieves this by using hands-on manipulatives and the use of similes, which the students encountered in previous learning processes. Even though the same similes exist in Maltese, *ħafif rixa* (light feather) and *tqil ċomb* (heavy lead), for *as light as a feather* and *as heavy as lead* respectively, similes seem to have been learnt primarily in the English language in this particular classroom (even in English lessons). A possible reason might be the emphasis of the *as... as* construction which denotes explicitly that something is being compared to something, which seems to be missing in conventional Maltese similes without the *like* element. After a student uses the English simile, as light as a feather, the teacher takes on the English word in her instruction to the students: *Araw hux light!* (Look whether it is really light!) in Turn 7. Nevertheless, soon afterwards she asks for the Maltese word of light in order to reassure that the students understood its meaning. Interestingly, afterwards the word *ħafifa* for light is increasingly used in discourse, especially the more the hands-on are used. Hence, for the teacher it is important that the students not only understand the meaning of the underlying concepts (in this case, the teacher controls it by requiring the students to activate Maltese explicitly, e.g., in Turn 8), but also be able to express the concepts in both Maltese and English, which can be interpreted as a content and language (in this case two languages) learning scenario. In contrast to the word *feather* (whose Maltese translation *rixa* was not mentioned), the teacher realized that the word *lead* was more difficult for the students to learn and use in discourse, hence she explicitly emphasizes its translation (in Turn 16). However, besides the use of simile, the teacher puts emphasis on mathematics learning with the ideas from everyday life. She recounts the use of lead for fishing in order to facilitate student understanding of its heaviness and its use in real world. Due to the fact that Maltese is the language in which the context of fishing was experienced (Maltese being the teacher's first language), she relies solely on Maltese to express the everyday idea for amplifying the learning of the concept of heaviness and simultaneously teach the students about fishing traditions in Malta. Hence, the teacher emphasizes the linking between words and their meanings in the first and second language, integrating language learning when teaching the content of heaviness and lightness.

The next transcript from the sixth-grade teacher introducing percentages shows how translanguaging can be used to facilitate mathematics learning:

Turn		Original	English translation
1	T2	(...) U xi tfisser il-kelma ninety percent, xi tfisser?	(...) What does the word ninety percent mean, what does it mean?
2	S2	Biċċa minn ħaġa.	A part from something.
3	T2	Brava. **Biċċa** minn ammont. Tafu xi tfisser per? Mela aħna għandna l-kelma percent. Percent magħmula minn żewġ kelmiet maqgħuda flimkien. (writes on whiteboard per-cent). Per-cent, magħmula minn kelma per. Xi tfisser il-kelma per?	Very good. A part from an amount. Do you know what per means? So we have the word Percent. Percent is made up of two words joint together. (writes on the whiteboard per-cent). Per-cent, made up of the word per. What does the word per mean?
4	S2	Żewġ affarijiet?	Two objects?
5	T2	Le, dik **pair**. That's the word **pair**. Pair equals two (writes down 2 = pair on the whiteboard). Issa aħna għandna **per**.	No, that is **pair**. That's the word **pair**. Pair equals two (writes down 2 = pair on the whiteboard). But we have **per**.
6	S4	Per, eżempju, jekk għandi lapes jiswa daqs il-lapes l-ieħor.	Per, for example, when I have one pencil and it costs the same as another pencil.
7	T4	Le, dik equivalent. Equal. Isa, iltqajna magħha l-kelma per.	No, that is equivalent. Equal. Well, we have already done the word per before.
8	S1	Għal kull wieħed?	For each one?
9	T2	**Għal kull wieħed**! Ngħidlek, I have two sweets **per** child. Xi tfisser?	**For each one**! I tell you, I have two sweets **per** child. What does it mean?
10	S3	Tnejn.	Two.
11	T2	Għandi żewġ ħelwiet **għal kull** tifel jew tifla. Two per child. Sewwa? U xi tfisser **cent**? Mil-liema kelma ġejja cent? Għidnija kemm il-darba din, iltqajna magħha f'**centi**meters. Iltqajna magħha kemm il-darba magħha x'ħin konna nagħmlu l-flus. Xi tfisser cent? Mil liema kelma għedtilkom li ġejja l-kelma cent, qisa?	I have two sweets **for each** boy or girl. Two per child. Ok? And what does cent mean? From which word does **cent** originate? We have said this very often, we have done this in **centi**meters. We have done it when we were learning money. What does cent mean? From which word, have I told you, comes the word cent, come on?
12	S5	Cento?	Cento? [Hundred in Italian]

13	T2	**Cento**! Li tfissir?	**Cento**! What does it mean?
14	S	(in choir) Hundred.	(in choir) Hundred.
15	T2	Hundred (confirming). Very good. Mela, for every hundred, hekk tfisser dik. Hekk tfisser. Percent, per cento, for every hundred. **Minn kull mija,** ngħidu bil-Malti. Mela, meta jiena għandi... ngħidlek forty... nista niktiba percent, u nista niktibha bħala symbol. Percent tinkiteb hekk. Forty percent. Forty percent nista niktiba b'mod differenti. Inti x'qetli? (to S2) Biċċa minn xi ħaga. Mela hija fraction. Vera jew le?	Hundred (confirming). Very good. So, it means for every hundred. That's what it means. Percent, per cento, for every hundred. From every hundred, we say in Maltese. So, when I have... I tell you forty... I can write it percent, and I can write it as a symbol. Percent can be written like this. I can write forty percent in another way. What have you just told me? (to S2) A part from something. This means, it is a fraction. Right or wrong?

Table 2: A transcript excerpt from the first phase of introducing the concept of percentages in Grade 6

In the above transcript excerpt (see Table 2), the terminology to be learnt in the lesson unit is clearly emphasized by the teacher in English, i.e. the concept of percentages. Prior to this transcript, the teacher asked the students where they had encountered the word percentages before, and a student replied that percentages are present in supermarket discounts in everyday life. At the beginning of the transcript, the teacher wants the students to understand where the word percent comes from, which she breaks down into per and cent. During this phase, she even activates a third language, Italian, for translation or repetition purposes: *per cento* which translates into for every hundred in order to facilitate student understanding of the whole (which in this case is always hundred). This example is typical for translingual teaching practices in which teachers make use of additional languages, which the students speak, in order to facilitate student learning. Further analyses of the transcript show that previously learnt mathematical concepts are retained cognitively in English, e.g. "equivalent" in Turn 7, however, the process of activating such concepts tends to be represented in Maltese, e.g. "iltqajna magħha l-kelma per" (We have already done the word per before) in Turn 7. This coincides with findings from the transcript analysis of the fourth-grade teacher, who uses the concept of similes and uses Maltese to activate student knowledge about the English similes required for learning the concept of weight.

It can be noticed that the translingual practices used by the teachers are not always in one direction, from the first to second language, but can also be vice versa, e.g. from English to Italian to Maltese, as in the case of the notion of for every hundred in the sixth-grade (see Table 2). Furthermore, the use of the languages by the teachers and students cannot be reduced merely to that of translation. For instance, when explaining the different modes of representation of percentages, the teacher uses English names for the mode names (to be written) and uses Maltese orally to support memorizing how to write them (see Turn 15). In another instance at the end of transcript in Table 2, it can be noticed that the teacher uses Maltese for argumentation, for linking the concepts of percentages and fractions: "Mela hija [forty percent] fraction" (This means, it [forty percent] is a fraction).

Concluding Remarks

From a mathematical education perspective, it is important to investigate how different languages can be used simultaneously to teach and learn mathematical concepts. Based on these short episodes about primary teachers introduce mathematical concepts using translanguaging, one can summarize that both Maltese and English can fulfill different functions in teaching and learning of mathematical content. Whereas Maltese seems to be more used for giving instructions to students to perform actions on hands-on or to activate previously learnt knowledge, especially from daily life experiences, English is used to verbalize more abstract ideas (as in the case of the concept of simile), and to store knowledge at the end of a learning process. These results provide an important foundation for future work in this case methodology study of investigating the use of Maltese and English among different teachers to teach mathematics.

References

Baker, C. (2011). *Foundations of Bilingual Education and Bilingualism* (3rd Edition). Buffalo: Multilingual Matters.

Garcia, O. (2009). Education, Multilingualism and Translanguaging in the 21st Century. In: A. Mohanty, M. Panda, R. Phillipson, & T. Skutnabb-Kangas (eds.), *Multilingual Education for Social Justice: Globalising the local*. New Delhi: Orient Blackswann, 128-145.

Hunter, M. (1982). *Mastery Teaching*. El Segundo, CA: TIP Publications.

Ministry for Education and Employment. 2014. *Mathematics - a revised Syllabus for Primary Schools.* Retrieved from https://curriculum.gov.mt/en/Curriculum/Year-1-to-6/Documents/pr_syllabi/syllab_pr_mathematics_yr1_to_yr6.pdf

Spagnolo, F., & Di Paola, B. (2010). European and Chinese cognitive styles and their impact on teaching mathematics. *Studies in Computational Intelligence, 277*, 1-267.

"The guided reinvention" in the teaching of mathematics at the tertiary level

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Abstract. The method of "the guided reinvention" (a kind of the genetic method) in the teaching of the advanced sections of abstract algebra including congruence relations in algebras (algebraic structures), quotient algebras modulo congruence relations, and homomorphic image theorems for algebras is considered.

Résumé : La méthode de la «réinvention guidée» (une sorte de méthode génétique) dans l'enseignement des sections avancées de l'algèbre abstraite, notamment les relations de congruence dans les algèbres (structures algébriques), les algèbres à quotient modulo, les relations de congruence et les théorèmes d'image homomorphes pour les algèbres est examinée .

The mathematical education developed in such a way that very little attention was given to methods of teaching the subject at the tertiary level (in universities and other institutions of higher education). Only since 1990-s the systematic research of methods of teaching advanced sections of mathematics began (Dubinsky et al., 1994; Tall, 1991). In the last decade papers devoted to the teaching of abstract algebra by the method of "the guided reinvention" appeared (guided reinvention) (Larsen, 2009; Larsen & Lockwood, 2013). The term "the guided reinvention" goes back to H. Freudenthal (1972), and in essence represents a genetic method (Safuanov, 1999). Elements of the genetic approach, in particular, open-approach, have been rather successfully applied in the teaching of elementary mathematics, for example, in the countries of Southeast Asia (Safuanov, 2015; Safuanov, Atanasyan, 2016; Safuanova & Safuanov, 2016).

In the above-mentioned papers on the teaching of abstract algebra by the method of "the guided reinvention" (Larsen, 2009; Larsen & Lockwood, 2013) the authors suggest to teach the most difficult concepts of groups, isomorphisms of groups, quotient groups etc. on the basis of essentially the sole example, namely the set of symmetries of a geometrical figure (an equilateral triangle or a square). In our opinion, this is not enough for the adequate formation of the concepts of group theory.

In our paper (Safuanov, 2005) the genetic approach in the teaching of a mathematical discipline is described. Its implementation requires two parts: 1) a preliminary analysis of the arrangement of the content and of methods of teaching and 2) the design of the process of teaching. The preliminary analysis consists of two stages: 1) the genetic elaboration of the subject matter and 2) the analysis of the arrangement of contents, the possible ways of representation, and the effect on students. The genetic elaboration of the subject matter, in turn, includes historical analysis of the subject. The purpose of the historical analysis is twofold: 1) to reveal paths of the origin of scientific knowledge that underlie the educational material and 2) to find out what problems generated the need for that knowledge and what were the real obstacles in the process of the construction of the knowledge.

Here is a short historical analysis for the concept of a group.

F. Klein, who had brought in the essential contribution to the development of the group theory due to "Erlangen program" of the study of geometry through the study of groups of geometrical transformations, argued that "the concept of a group was originally developed in the theory of algebraic equations" (Klein, 1989, p. 372). Thus, groups, in his opinion, have arisen as groups of permutations. However, such fundamental concept as a group had also other roots in mathematics. As indicated in "The Mathematical encyclopedic dictionary" (1988, p. 167), sources of the concept of a group are in the theory of solving algebraic equations as well as in geometry, where groups of geometrical transformations have been investigated since the middle of the 19-th century by A. Cayley, and in number theory, where in 1761 L. Euler "in essence used congruences and partitions into congruence classes, that in the group-theoretic language means decomposition of a group into cosets of a subgroup" (ibid.). However, abstract groups were introduced by S. Lie only at the end of the 19-th century.

The main conclusion from this historical analysis is that the theory of groups has grown out of the development of many diverse ideas and constructions in mathematics and serves to the generalization and more effective theoretical consideration of these ideas and constructions.

Historical analysis implies that the preliminary study of the wider range of elementary examples is required: not only groups of symmetries, but also groups of (integer, rational, real) numbers under the addition and multiplication, and also groups of congruence classes modulo natural numbers greater than 1 (Safuanov, 2005; Safuanov, 2009). On the basis of the wide range of examples, it is possible to build problem situations, and also to consider various applications of the concepts under consideration. This will also help to increase students' motivation.

We need also the historical analysis of equivalence relations (Safuanov, 2005).

Equivalence relations appeared for the first time in the educational literature in 1930 (Van der Waerden, 1930) as a generalization of the relation of equality. There, partitions into cosets with an example of the partition into classes of even and odd numbers were introduced. Note that in the book "Foundations of mathematics" (published in 1934) written by D. Hilbert and P. Bernays, the term 'equivalence relation' was not yet used, and it was spoken only about as 'some sort of coincidence' (Hilbert & Bernays, 1982, p. 213). Historically, the roots of equivalence relations and partitions into cosets can be traced as early as works of Euler, who essentially considered congruence relations modulo positive integers and partitions into remainder classes.

In Soviet undergraduate textbooks equivalence relations only appeared at the end of 1970s, (Kulikov, 1979; Kostrikin, 1977).

In this paper we propose "the guided reinvention" of the concepts of normal subgroups, quotient groups, ideals in rings, and of quotient rings at the deeper level of learning abstract algebra, for example, in a master program.

Based on the students' (although fragmentary) knowledge of the concepts of sets, maps, groups, subgroups, rings, subrings, group and ring homomorphisms etc., we can develop a general concept of an algebra (algebraic structure), i.e. a non-empty set with algebraic operations of various ranks given on it, and also of an algebra homomorphism. After this, our next task is to formulate the definition of a congruence relation on an algebra. This is the most difficult concept. It is very difficult to construct the adequate problem situation in order to prepare students to "reinvent" this concept. The properties of congruences modulo integer $m>1$ connected with addition and multiplication can serve as the best motivating example. We think that it would take extremely long time to wait for the "reinvention" in this case. It would be better if the lecturer would formulate the definition and support it by examples (congruences modulo an integer greater than 1, parity of permutations etc.). Furthermore, we can introduce the concept of a quotient algebra modulo the congruence relation, of a natural homomorphism, and state and prove the general homomorphic theorem for algebras.

Returning to groups, we suggest students to consider the concept of a congruence relation in a group, to investigate the quotient group modulo given congruence relation, and to find the class of a neutral element modulo given congruence relation. It is not difficult to show that this class is a subgroup and, moreover, a normal one, and all the congruence classes are cosets (simultaneously left and right ones) of it. Thus, at the deeper level of consideration, the necessity of the normality of a subgroup for constructing a quotient group can be obtained even in a simpler way than by usual methods in the main undergraduate courses on abstract algebra.

Similarly, one can study congruence relations in a ring. Having considered the class of zero modulo given congruence relation, it is not difficult to show that it is an additive subgroup of a ring closed under both the left- and right-multiplication by elements of the ring. Thus, this class turns out to be a (two-sided) ideal. The quotient ring modulo this congruence relation coincides with the quotient ring modulo given ideal. Again, it is useful to illustrate this consideration by the example of the ring of integers and the relation of congruence modulo an integer greater than 1, which is a congruence relation here.

Thus, it is possible to carry out "the guided reinvention" of concepts, already familiar to students to some extent, at the more advanced level of the study of the subject, thereby consolidating and deepening the understanding of the most complex mathematical concepts and ideas.

The experience of implementation of the described approach in the teaching of the course of algebra for students of a master program for prospective mathematics teachers, demonstrated the increased interest in the subject, the successful mastering of the meaning and application of homomorphic image theorems for groups and rings. Moreover, after attending the course, the students became able to independently solve difficult problems requiring the application of the homomorphic image theorem.

References

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. – *Educational studies in Mathematics,* No. 27 (3), pp. 267-305.

Freudenthal, Hans (1973), *Mathematics as an Educational Task*. – Dordrecht: Reidel.

Hilbert, D., Bernays, P. (1982). *Foundations of Mathematics*. Moscow: Nauka (Russian).

Klein, F. (1987). *Elementary Mathematics from the Higher Point of View* (Elementarnaya matematika s tochki zreniya vysshey). Moscow: Nauka (In Russian).

Kostrikin, A.I. (1977). *Introduction to Algebra*. Moscow: Nauka (Russian).

Kulikov, L. Ya. (1979). *Algebra and Number Theory*. Moscow: Vysshaya Shkola (Russian).

Larsen, S. (2009). Reinventing the concepts of the group and isomorphism: The case of Jessica and Sandra. – *Journal of Mathematical Behavio*r, 28 (2-3), pp. 119-137.

Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. – *Journal of Mathematical Behavior*, 32(4), pp. 726–742.

Safuanov, I. (2009). Design of a system of teaching elements of group theory. – *Symmetry: Culture and Science,* v. 20, No 1-4, pp. 361–370.

Safuanov, I. S. (1999). *Theory and practice of teaching mathematical disciplines in pedagogical institutes (Teoriya i praktika prepodavaniya matematicheskikh distsiplin v pedagogicheskikh institutakh).* – Ufa: Magrifat.

Safuanov, I. S. (2015). An open approach to teaching mathematics (Otkrytyj podkhod k obucheniyu matematike). – *Proceedings of the 1st All-Russian Scientific and Practical Conference "Universities in the system of search and support of mathematically gifted children and youth"*. – Maykop: Adygeh State University Publishers, pp. 128-133.

Safuanov, I.S. & Atanasyan, S. L. (2016). Mathematical Education in Singapore: Traditions and Innovations (Matematicheskoe obrazovaniye v Singapure: traditsii I innovatsii). – *Nauka I Shkola,* No. 3, pp. 38-44.

Safuanov, I.S. (2005). The genetic approach to the teaching of algebra at universities. –

International Journal of Mathematical Education in Science and Technology, v. 36. No. 2-3, pp. 255-268.

Safuanova, A.M. & Safuanov, I.S. (2016). "Open approach" and "Lessons study" – ways of improving mathematical education ("Otkrytyj podkhod i "Issledovaniye urokov" – puti sovershenstvovaniya matematicheskogo obrazovaniya). – *Nizhegorodskoe obrazivanie,* No. 2, pp. 146-150.

The Mathematical Encyclopedic Dictionary (Matematicheskij entsiklopedicheskij slovar') (1988). Moscow: Sovetskaya Encyclopedia (In Russian).

Van der Waerden, B.L. (1930). *Moderne Algebra*. Berlin: Springer.

Trous noirs numériques: π et e en dialogue permanent avec la realité en l´histoire des mathématiques

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Resume: Ainsi que un trou noir est un corps avec une gravité si forte que rien ne peut y échapper, même pas la lumière, il y a aussi des nombres qui attirent les autres lors de certaines opérations.
Il y a des nombres qui, à travers l'histoire, sont devenus énormément célèbres pour diverses raisons. Dans certains cas, sa renommée est simplement due à des coïncidences historiques; parfois il apparaît au centre de certaines découvertes scientifiques très pertinentes.
Les nombres π [1] et e sont intervenus différemment tout au long de l'histoire de l'humanité dans les problèmes de réalité géométrique et aussi dans la résolution des problèmes de la vie réelle.
Dans cet article, il vise à révéler l'omniprésence des nombres ***π et e***, et à tenter de répondre à la question: l'ubiquité [3] de ces nombres est-elle une propriété des phénomènes où ils apparaissent ou ne se réfère qu'au langage utilisé pour sa description?

1. Le Monde des nombres

Le monde des nombres est sans fin, infini. Approfondir devient plus compliqué au fur et à mesure que vous avancez. Ainsi naît et se développe la théorie des nombres, qui aujourd'hui [5], [6] est une branche forte et cohérente des mathématiques, il y a des nombres amis, des cousins, abondants, transcendants, rationnels, aléatoires, normaux, réels, complexe, apocalyptique, calculable, trous noirs numériques, ... [2]

Pourquoi tant de fascination pour les nombres?

Les nombres les plus importants dans l'histoire des mathématiques et de la physique sont connus sous le nom de lettres. Par exemple, je citerai certains comme:

-Le numéro d'Avogadro est N_A, *et* a de la valeur $6.022140857(74)\times10^{23}$ mol^{-1}

-La constante cosmologique de Hubble, H_0 est 21'9, et vient exprimée en (Km/s)/Megas Année Lumiére.

-La constante de Planck, h, est une constante physique qui joue un rôle central dans la théorie de la mécanique quantique et prend le nom de son découvreur, Max Planck, un des pères de cette théorie. C'est la constante souvent définie comme la quantité élémentaire d'action et a la valeur de 6.62606957x10-34 juillet/ seconde.

-L'unité imaginaire $i=\sqrt{-1}$ peut être utilisée pour étendre formellement la racine carrée des nombres négatifs, confirmant ainsi le théorème fondamental de l'algèbre. En physique quantique, l'unité imaginaire simplifie la description mathématique des états quantiques variables dans le temps.

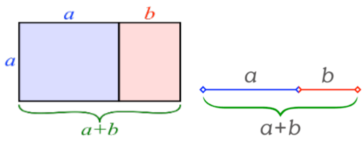
-Grâce à la formule de De Moivre:

$$[\cos(x)+isen(x)]^n=\cos(nx)+isen(nx)$$

les logarithmes des nombres négatifs sont également exprimés pour l´unité ***i***, donc

$$\ln(-1)=i\pi$$

-Le nombre d'or, ***φ*,** nombre algébrique irrationnel qui a beaucoup de propriétés intéressantes et qui a été découvert non pas comme une expression arithmétique, mais comme une construction géométrique.



$$\frac{a+b}{a}=\frac{a}{b}=\varphi$$

-Les nombres ***π* *et e*** sont importants en mathématiques dans le contexte de certains processus répétitifs qui donnent lieu à des résultats qui ne varient plus dans les itérations successives. Tout comme un trou noir est un corps avec une gravité si forte que rien ne peut y échapper, même pas la lumière, il y a aussi des nombres qui attirent les autres lors de certaines opérations, et ils ne sont pas l´exception.

2. Nombre *π*

2.1. Histoire

Le nombre ***π*** fait partie de la culture humaine et de l'imagination. Il a été étudié pendant plus de vingt-cinq siècles (2500 ans). Sa popularité énorme découle de son utilisation fréquente dans les mathématiques, la physique et l'ingénierie. Si nous considérons l'ingénierie comme l'un des moteurs du monde, nous pouvons comprendre que le nombre ***π*** est fondamental pour notre développement en tant que société. Le nombre ***π*** est la figure la plus étudiée et admirée de l'histoire, et il y a tellement d'informations, tant de littérature sur ***π*** que dans ce travail nous voulons mettre en évidence l'importance et l'application dans certains aspects de la vie réelle.
Sa "majesté", le nombre π, le nombre réel le plus emblématique de nombres irrationnels, naît d'une proportion géométrique simple et millénaire: c'est le rapport constant, P / D = π, entre le périmètre P de toute circonférence et son diamètre D.

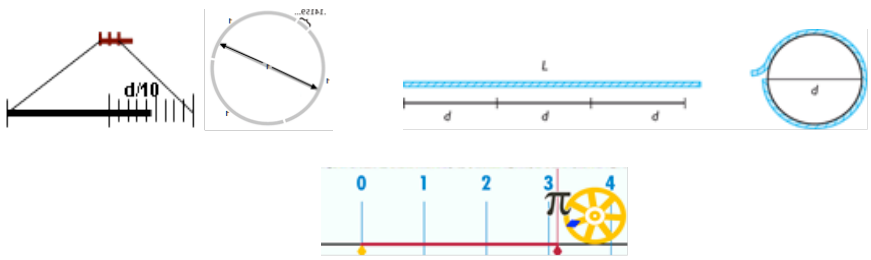


Fig.1. Nombre ***π***

Et comme toutes les cultures ont été des consommateurs de cercles, l'intérêt pour ***π*** a toujours été présent et encore joue un rôle important. Nous l'avons présent dans des pots, des roues, des verres, des bouteilles, des biscuits, des balles, des balles, des montres ... Là où il y a des cercles ou des sphères, il y a ***π*** et donc dans les mesures associées.
La première référence connue date d'environ 1650 a. J.C. L'histoire est pleine de tentatives pour approcher la valeur réelle du nombre π au moyen de polygones. La controverse bien connue sur le Calculus était une discussion entre les mathématiciens du dix-septième siècle, Isaac Newton et Gottfried Leibniz (principalement entretenus par ses disciples), sur lesquels était celui qui avait inventé le calcul. Ce conflit a commencé à émerger vers 1699 et a éclaté avec une grande force. Sans doute domestiqué l'infini, ils ont emmené les mathématiciens au paradis et leur ont appris comment passer du fini au infini, on peut dire avec eux que le

scandale est arrivé. Le calcul n'est plus une simple question de mesure des polygones, mais est maintenant une question mathématique plus profonde.
Par exemple, Newton en 1665 dans le feu de Londres découvre la série avec laquelle calcule 16 chiffres exacts

$$\pi = \frac{3\sqrt{3}}{4} + 24(\frac{1}{12} - \frac{1}{5.2^5} - \frac{1}{28.2^7} - \frac{1}{72.2^9} - \ldots)$$

Abraham Sharp (1651-174), avec la formule

$$\pi = \sum_{0}^{\infty} \frac{2(-1)^k . 3^{\frac{1}{2}-k}}{2k+1}$$

Il a obtenue 71 decimauxs corrects pour le nombre **π.**
John Machim (1686-1751), astronome de la Royal Society a obtenu cent chiffres avec la formule qui porte son nom:

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

Joham Martin Zacharias Dase (1824-1861) occupe une des places importantes dans l'histoire des mathématiques. Avec l´expression

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8})$$

Il a obtenue en 1844, 200 decimauxs. L' incroyable est qu'il a obtenue avec la mémoire (il a été considéré comme "Zacharias version humaine d'un ordinateur").
En 1882, l'allemand Lindemann jeta une cruche d'eau froide sur les creuseurs enthousiastes en prouvant que π n'est pas un nombre algébrique et n'est donc pas constructible. C'est lui qui a montré sa transcendance. De cette façon, il est sorti complètement de son cadre géométrique.
Il a conclu qu'une combinaison linéaire de puissance de "e" et de coefficients algébriques (réels ou complexes)

$$A_1 e^{B_1} + A_2 e^{B_2} + \ldots + A_{n-1} e^{B_{n-1}} + A_n e^{B_n} = 0$$

c'était impossible (sauf pour tous les coefficients $A_k = 0$), comme la formule bien connue d'Euler

$$e^{i\pi} + 1 = 0$$

De là, il est expulsé du paradis de l'inconnu. Avant Lindemann, on savait que la transcendantalité impliquait ¡**l'impossibilité de la quadrature du cercle**!

2.2. Applications
Le nombre **π** est un nombre clé en mathématiques [7]. Un nombre qui a de nombreuses et incroyables applications dans la vie réelle. On peut se demander: les gens connaissent-ils vraiment de telles applications en dehors de ce qu'ils nous ont enseigné à l'école que **π** est la relation entre la circonférence d'un cercle et la longueur de son diamètre? [8]
¡ **π** est plus que juste un nombre qui nous a forcés à étudier en mathématiques!
Ses applications les plus directes sont dans les calculs de la superficie et du périmètre d'un cercle, ainsi que pour le volume d'un cylindre**.**

π est connu depuis près de 4000 ans et a servi dans l'antiquité pour la construction des pyramides.


Fig.2. Pyramides d'Egypte

Mais ***π*** [14] a aussi d'importantes applications pratiques en informatique, en astronomie, en économie ou en physique, entre autres disciplines [9].
a) La vitesse des ordinateurs est testée en leur faisant calculer ***π***, parmi ces utilisations il y a tout calcul dans lequel il y a des cercles, comme l'orbite des satellites.
b) Il est également utile d'étudier les courbes. Ainsi, ***π*** aide à comprendre les systèmes périodiques ou oscillants, tels que les horloges, les ondes électromagnétiques et même la musique.
d) Dans Statistics, ***π*** est utilisé pour calculer l'aire sous une courbe de distribution, qui est applicable pour connaître la distribution des scores standardisés, des modèles financiers ou des marges d'erreur dans les résultats scientifiques.
e) En outre, il est utilisé dans des expériences de physique des particules, telles que celles qui utilisent le Large Hadron Collider. Les scientifiques ont utilisé ***π*** pour démontrer la notion trompeuse que la lumière fonctionne comme une particule et une onde électromagnétique et, ce qui est plus impressionnant, pour calculer la densité de tous les nombres IRRATIONNELS [17].
f) Que la planète Terre est pratiquement sphérique, tout comme de nombreuses trajectoires spatiales sont des ellipses, elle donne aussi la notoriété dans l'Univers. Mais les apparences de ***π*** [10] se retrouvent aussi dans toutes sortes de formules ou de modèles mathématiques: il joue un rôle clé en trigonométrie (et donc en topographie, géodésie ou navigation), en calcul, en distributions statistiques (en cloche), dans les résultats de probabilité, dans les équations fondamentales de la physique (principe d'incertitude de Heisenberg, équation de champ d'Einstein, loi de Coulomb, troisième loi de Kepler ...)

3. Número *e* [19]

3.1. Histoire

L'histoire de ***π*** a été largement dit, certainement pas seulement parce que son histoire est amené des temps anciens, mais aussi parce qu'une grande partie de celui-ci peut être compris sans une connaissance approfondie des mathématiques [11]. La constante mathématique ***"e"*** [4] est l'un des nombres irrationnels les plus importants. Le nombre ***"e"*** apparaît en mathématiques peut-être trente siècles après ***π***. Cependant, c'est un nombre qui se produit naturellement dans beaucoup de considérations mathématiques. Il est approximativement égal à 2,71828 et apparaît dans diverses branches des mathématiques, étant la base des logarithmes naturels et faisant partie des équations à intérêts composés et de nombreux autres problèmes.
Les premières références à la constante ***"e"*** [16] ont été publiées en 1618 dans le tableau d'une annexe d'un travail sur les logarithmes de John Napier. Cependant, cette table ne

contenait pas la valeur de la constante, mais était simplement une liste de logarithmes naturels calculés à partir de cette constante.

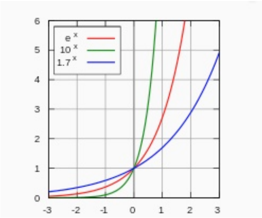


Fig.3. Nombre ***e***

On croit que la table a été écrite par William Oughtred. Quelques années plus tard, en 1624, le nombre ***"e"*** [18] est de nouveau impliqué dans la littérature mathématique, mais pas entièrement. Cette année-là, Briggs a donné une approximation numérique des logarithmes en base 10, mais n'a pas mentionné explicitement le nombre ***e*** dans son travail. Cependant, peut-être de façon inattendue, et non par logarithmes ***e*** est découvert, mais l'étude du problème de l'intérêt composé adressé par Jacob .Bernoulli en 1683. Leonhard Euler a popularisé l'utilisation de la lettre ***e*** représenter la constante; Il était aussi le découvreur de nombreuses propriétés qui s'y rapportaient. Il a fait plusieurs contributions par rapport à ***e*** dans les années suivantes, mais ce n'est qu'en 1748 quand il publie son Introductio in Analysin infinitorum qu'il donne un traitement définitif aux idées sur ***e.*** Là, il a montré que

$$\boldsymbol{e} = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

et lui a donné une approximation de 18 décimales:

e = 2,718281828459045235 sans montrer comment il l'a eu.

Le nombre "e" ainsi que le "π" et le nombre d'or "φ", est un nombre irrationnel, non exprimable par un rapport de deux nombres entiers; ou, il ne peut pas être représenté par un nombre décimal exact ou un nombre décimal périodique. De plus, comme "π" est un nombre transcendant, c'est-à-dire qu'il ne peut être la racine d'une équation algébrique avec des coefficients rationnels. La valeur de "e", tronquée à ses premières décimales, est la suivante: 2.71828182845904523536 ...

Leonhard Euler a popularisé l'utilisation de la lettre "e" pour représenter la constante; Il a également découvert de nombreuses propriétés qui s'y rapportent.

Contrairement à "π", l'introduction du nombre "e" en mathématiques est relativement récente, ce qui a du sens si l'on considère que cette dernière a une origine analytique et non géométrique, comme la première. Dans les mots du mathématicien israélien Eli Maor: *"The story of "**π**" has been extensively told, no doubt because its history goes back to ancient times, but also because much of it can be grasped without a knowledge of advanced mathematics. Perhaps no book did better than Petr Beckmann's A History of **π**, a model of popular yet clear and precise exposition. The number e fared less well. Not only is it of more modern vintage, but its history is closely associated with the calculus, the subject that is traditionally regarded as the gate to higher mathematics"."*

Fig. 4. *e* l'historie d' un nombre

*"L'histoire de "π" a été largement racontée, sans doute parce que son histoire remonte à l'Antiquité, mais aussi parce qu'une grande partie peut être comprise sans une connaissance des mathématiques avancées. Peut-être aucun livre ne valait mieux que A History of π de Petr Beckmann, un modèle d'exposition populaire mais clair et précis. Le nombre "e" a si bien réussi. Non seulement est-ce un âge plus moderne, mais son histoire est étroitement liée au calcul, le thème traditionnellement considéré comme la passerelle vers les mathématiques supérieures***".**

3.2. Applications

Essayons de demander:
Qu'ont-ils en commun, une toile d'araignée, des lignes électriques, l'âge d'un fossile, l'intérêt d'un compte en banque ou la croissance d'une population de bactéries? [15]

Fig. 5. Examples du nombre *e*

Au-delà de sa beauté mathématique, le nombre ***e*** a des implications importantes dans le monde que nous connaissons. Nous exposerons seulement quelques uns:

(a) En ***Biolog***ie, par exemple, l'une de ses principales applications est la croissance exponentielle.
(b) Lors de la ***datation d'un fossile***, la constante d'Euler est également présente.
(c). La ***Médecine légale***, comme les paléontologues, doit également tenir compte de ce nombre. Et est ce *"e"* permet de déterminer dans un meurtre le moment de la mort. Pour cela, il est nécessaire d'appliquer la loi de Newton sur le refroidissement qui établit la vitesse à laquelle un corps se refroidit.
(g) En ***Mathématiques financières***, il est utilisé pour calculer les intérêts continus:
(h) En ***Ingénierie***, lorsqu'une chaîne ou un câble est accroché par les extrémités, il tend à adopter une forme liée au nombre *"e"*.
(i) Dans les ***êtres vivants***, il existe des courbes liées au nombre e. L'un d'eux est la colonne vertébrale logarithmique.
(j) ***Fractales*** [12]. Dans la nature, il existe de nombreux exemples de façons appartenant à la géométrie euclidienne (hexagones, cubes, tétraèdres, carrés, triangles, etc.), mais produit aussi des objets vastes de la diversité qui contournent la description euclidienne, par exemple, des formes dessinées sur la Terre, vie naturelle, dans la nature, ...

Deux opinions: (1) Le peintre Paul Cézanne: "*Tout dans la nature peut être vu en termes de cônes, de cylindres et de sphères*". C'est une déclaration programmatique en référence à son style pictural et vient à nos cheveux comme une description d'une vision euclidienne de la nature. (2) La réponse serait mis Mandelbrot répondre: "*Les nuages ne sont pas des sphères, les montagnes ne sont pas des cônes, les côtes ne sont pas des cercles, des troncs d'arbres ne sont pas lisses et rien que la lumière se déplace en ligne droite*". Si le message de Mandelbrot est que la Nature répond mieux à un autre type de description, il serait commode de le prouver au-delà de la simple intuition.

Fig.6. Exemples du nombre e de fractalité

Dans ces cas, les fractales nous fournissent un meilleur moyen d'explication, et «le nombre e fait une présence permanente» [15] lorsque nous avons besoin de mesurer.
Pour la ligne:

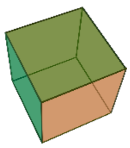
$$\mathbf{D} = -\frac{\ln 2}{ln(\frac{1}{2})} = 1$$

Pour le carré

$$\mathbf{D} = -\frac{\ln 4}{ln(\frac{1}{2})} = 2$$

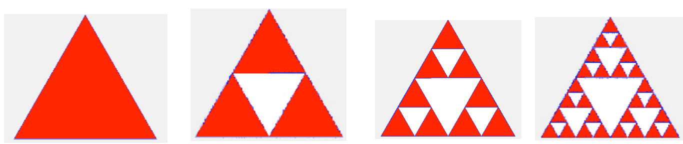
Pour le cube:

$$\mathbf{D} = -\frac{\ln 8}{ln(\frac{1}{2})} = 3$$



Pour le Triangle Sierpinski:

$$\mathbf{D} = -\frac{ln3}{ln(\frac{1}{2})} = 1,589496$$



4. Conclusion

Dans cet article, nous avons utilisé les nombres **π** et ***e*** pour:

a) Approfondir dans les propres méthodes d'investigation en mathématiques: la particularización, la recherche de lois générales, la construction de modèles [13], la généralisation, l'utilisation d'analogies, de conjectures et de démonstrations.

b) Utiliser des modèles mathématiques [13] avec la présence des nombres π et e pour la mathématisation de la réalité et la résolution des problèmes, expérimenter leur validité et leur utilité, critiquer leurs limites, les améliorer et communiquer leurs résultats et conclusions

Avec tout cela, nous approchons nos étudiants à la connaissance mathématique priorisant l'approche et la résolution des défis [11], la recherche de modèles explicatifs, l'enquête et la découvert

5. Références

[1] Beckmann, P. (1993) *A History of Pi*. Ed. Barnes and Books, New York

[2] Davenport, Harold (1999). *The Higher Arithmetic: An Introduction to the Theory of Numbers* (7.ª edición). Cambridge, England: Cambridge University Press.

[3] Duperret, J.C. (2009). *De la modélisation du monde au monde des modèles. Le délicat rapport "mathématiques –réalité*". *Bulletin 484.* APMEP.pp. 648-650.

[4] Euler, Leonhard (1921). *Opera Omnia, Series Prima, Vol. 6: Commentationes Algebraicae*. Leipzig: Teubner. pp. 350-369

[5] Izquierdo, M., Espinet, M., García, M.P., Pujol, R.M., & Sanmartí, N. (1999). *Caracterización y fundamentación de la ciencia escolar. Enseñanza de las Ciencias*. Núm. extra junio, 79-92.

[6] Izquierdo, M.; Espinet, M.; Bonil, J.; Pujol, R.M. (2004). Ciencia escolar y complejidad. Investigación en la escuela, 53, 21-29.pp.25

[7] Jami,C. *Une histoire chinoise du 'nombre π'.* Archive for History of Exact Sciences 38 (1988), pp.39-50.

[8] Knuth, Donald E. (4 de julio de 1997). *The Art of Computer Programming* **I**. Addison-Wesley Professional. p. 183.

[9] Lehmann, I.; Posamentier, A.S. (2006). *Pi: A Biography of the World's Most Mysterious Number.* Pp.89-97.Ed. Orient Black Swan Universities Pess

[10] Posamentier, A.S; Lehmann, I. (2006). *La proporción trascendental***.** pp.120-135 Ed. (Ariel)

[11] Romero , S. (2011). *La resolución de problemas como herramienta para la modelización matemática.* Review Modelling in Science Education and Learning: MSE*L.* Vol. 4.pp. 35-70.

[12] Romero, S. (2013). *Sistemas Dinámicos. Fractales*. CIBEM-VII. Video Conferencia *Plan CEIBAL-* 2013.Uruguay.

[13] Romero, S., Rodríguez, I.M., Benitez, R., Romero, J., & Salas, I.M. (2015). *Problem solving as instruments for the mathematical modeling: Examples For Real life*. Review Modelling in Science Education and Learning*: MSEL.* Vol. 8(2).pp. 51-65.

[14] Romero S. (2018). *Con **π** al fin del mundo. Día del número Pi* (14-03-2018) http://huelvaya.es/2018/03/14/la-uhu-se-suma-a-la-celebracion-del-dia-del-numero-pi/

[15] Romero S. (2018). *El número-e-Agujero Negro Numérico*. http://www.uhu.es/etsi/numero-e/doc/Actividad_Dia_Numero_E.pdf. 07-02-2018. Escuela Técnica Superior de Ingeniería-Universidad de Huelva

Web Site:

[16] http://eulerarchive.maa.org//docs/originals/E532.pdf

[17] https://arxiv.org/ftp/arxiv/papers/0704/0704.1282.pdf

[18] http://www-history.mcs.st-andrews.ac.uk/HistTopics/e.html
[19] https://es.wikipedia.org/wiki/Número_e

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Piaget's Legacy: What is Reflecting Abstraction?

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Abstract : In its classical Lockean sense, the concept of abstraction refers to a two-sided process, in which something is retained, while at the same time something else is left out. Therefore, every abstraction presupposes a certain kind of material or substance upon which it is carried out. Depending on the material upon which an abstraction is exercised, Piaget distinguishes between empirical and reflecting abstractions. In this paper, the notion of reflecting abstraction is reconstructed in three steps: Firstly, the shortcomings of the classical theory of abstraction in generally explaining the formation of concepts are developed; secondly, it is shown in how far these shortcomings can be overcome by modeling certain processes of concept formation as reflecting abstractions; thirdly, the explanatory power of the notion for the formation of mathematical concepts is exemplified in the case of what Freudenthal has called ›counting number‹.

Résumé : Dans son sens classique de Lockean, le concept d'abstraction se réfère à un processus à deux côtés, dans lequel quelque chose est retenu, tandis qu'en même temps quelque chose d'autre est omis. Par conséquent, toute abstraction présuppose une certaine sorte de matière ou de substance sur laquelle elle est exécutée. Selon le matériau sur lequel s'exerce une abstraction, Piaget distingue les abstractions empiriques et réfléchies. Dans cet article, la notion d'abstraction réfléchissante est reconstruite en trois étapes: Premièrement, les insuffisances de la théorie classique de l'abstraction dans l'explication générale de la formation des concepts sont développées; deuxièmement, on montre dans quelle mesure ces lacunes peuvent être surmontées en modélisant certains processus de formation de concepts comme des abstractions réfléchissantes; troisièmement, le pouvoir explicatif de la notion de formation de concepts mathématiques est exemplifié dans le cas de ce que Freudenthal a appelé ›le nombre de comptage‹.

Introduction : What is Abstraction?

In its classical Lockean sense, the concept of abstraction refers to a two-sided process, in which something is *retained*, while at the same time something else is *left out*: »[T]he mind«, Locke writes, »makes the particular ideas received from particular objects to become general […] by considering them as they are in the mind such appearances,—separate from all other existences, and the circumstances of real existence, as time, place, or any other concomitant ideas. This is called abstraction, whereby ideas taken from particular beings become general representatives of all of the same kind; and their names general names, applicable to whatever exists conformable to such abstract ideas« (Locke, 1836, book 2, chapter 11, §9).

This description allows us to identify a key characteristic of the process of abstraction: Every abstraction presupposes a certain kind of material or substance upon which it is carried out. In other words, we cannot simply abstract something, but rather we always abstract *from* something. For Locke, this substance is given by one or several particular ideas *inside* our mind that we have received *qua* sensual experience from objects *outside* of our minds. By comparing these particular ideas with each other and focusing our attention on certain similar attributes, while at the same time neglecting all other attributes, in which these particular ideas may differ from each other, these particular ideas are transformed into general ideas. A general idea thus differs from a particular idea in the sense that it is »capable of representing more individuals than one« (Locke, 1836, book 3, chapter 3, §6). The word ›apple‹, for example, does not refer to a particular individual, but it rather can be applied to a whole class of individuals. Whenever we stumble over an object in our stream of experience that possesses the set of attributes that are characteristic of the concept of apple (e.g. a spheric shape; a stem; red, yellow, green or sometimes pink in color; yellowish-white flesh with a core, a certain taste, and so on), then we are able to identify this object as an apple, even though we might have never seen this particular being ever in our lives before. By means of general ideas or concepts, we reduce the complexity of our environment, are able to build up stable expectations, and thus attenuate the necessity of permanent learning (Bruner, Goodnow & Austin, 1986, p. 11-15).[1] Although our environment is constantly changing, so that everything that surrounds us is actually always new, we are, at least, able to treat new environmental situations as particular instances of a general idea that we have already acquired before. Due to their massive range of application, the pressing question obviously is to further clarify the genesis of these general ideas. It was Locke's conviction that *all general ideas are made by the one form of abstraction* described above and that, therefore, *all general ideas descend in one way or another from particular ideas that are given to us by our senses* (cf. Locke, 1836, book 2, chapter 12, §1).

›Apple‹ and ›Apples‹

If we try to provide a model of how a child could acquire the concept associated with the word ›apple‹, Locke's description of the process of abstraction seems to be in working order: As being part of a language community, the child will sooner or later be confronted with situations in which he or she hears phrases such as ›This is an apple!‹ or ›No, this is not an apple, but a pear!‹. These phrases are probably accompanied by ostensive gestures, where the speaker points at the instances and non-instances of the concept as they present themselves in his or her sensory experience. From these ›shared apple experiences‹ the child then 1) must distinguish in his or her own stream of experience the chunks referred to as instances of apples, 2) recognize several of those chunks as different instances of one and the same kind and 3) retain those attributes that all of the particular instances have in common, while leaving out all other attributes, in which the particular instances differ from each other. Here, it is quite persuading that the substance of abstraction is given to the child by elements of his or her sensory experience. It is the set of particular ideas, the concrete apple experiences, from which the child abstracts to build the general idea of an apple.

Let us now consider another, slightly more complex, example and have a look at the concept associated with the plural, that is, the concept associated with the word ›apples‹: At first sight, one might think that the sensory material from which this general idea is abstracted must be a collection of apples, or, more precisely, more than one apple, and from there the abstraction machinery runs smoothly. But that is an illusion. Although, it is hardly imaginable how we could ever come to the concept of apples without experiencing more than one apple, we immediately get into trouble as soon as we try to specify which part of our sensory experience it exactly is, which all these collections of apples have in common. How is it even possible to recognize collections of various quantities and spatial configurations as different instances of one and the same kind? Or, to come back to the first step: Is it not also possible to say ›I have apples at home‹ if one apple can be found in the kitchen, while another one is in my room? In other words, the plural is also applicable to situations in which it is physically impossible to

[1] In this paper, I will use the words ›general idea‹ and ›concept‹ synonymously.

experience both apples at the same time. This observation leads us to a promising trace: If it is no necessary condition that we can actually experience all members of the collection of apples, then the substance or material for our process of abstraction cannot be the concrete sensory material but must lie somewhere else. So, Locke's theory of abstraction, which happened to work in the case of the concept of apple, breaks down in general, and it is this very moment, where Piaget's concept of reflecting abstraction comes into play.

Empirical and Reflecting Abstraction

Piaget retained Locke's conviction that all concepts are made by processes of abstraction, but he extended the conceptual scope of abstraction by giving up the idea that the substance or material of abstraction solely lies in the concrete sensory material. At any level of the conceptual development, there are, for Piaget, always two possibilities for the substances upon which abstractions are exercised.[2] The first possibility is more or less congruent with Locke's theory of abstraction: »At the sensorimotor levels, empirical abstraction draws information from objects and from the material and observable characteristics of actions« (Piaget, 2001, p. 317). This is what Piaget calls *empirical abstraction*. The second possibility, and this is the extension in comparison to Locke, is labeled by Piaget as *reflecting abstraction* because it refers to a process where the mind reflects upon its own operations. It is the mind's own operations that form the material for the processes of abstraction: »[W]hen we are acting upon an object, we can also take into account the action itself, or operation if you will, since the transformation can be carried out mentally. In this hypothesis the abstraction is drawn not from the object that is acted upon, but from the action itself« (Piaget, 1971, p. 16). The idea that abstraction refers to a two-sided process, in which something is *retained*, while at the same time something else is *left out* can thus be maintained because the difference between empirical and reflecting abstraction only lies in the ›something‹ upon which the abstraction is exercised. The difference is located in the *content* and not in the *form* of abstraction: »Empirical abstraction runs over observables, reflecting abstraction ranges over coordinations« (Piaget, 2001, p. 317)

On the basis of this distinction, we are now able to resolve the problems we have faced in our attempt to model the formation of the concept of plural as an empirical abstraction. What is common in all situations where the plural ›apples‹ is applicable cannot be found at the level of the sensory material (Piaget's ›observables‹) but, instead, it lies at the level of the child's mental operations upon sensory material (Piaget's ›coordinations‹). To move from ›apple‹ to ›apples‹, the child must reflect upon its own mental operations. It must become aware of the fact, that it *repeats* the act of identifying certain chunks of its sensory experience as apples. By performing a reflecting abstraction, that is, by solely focusing its attention on the *sameness* of the acts of identification (›Hello! Apple again!‹) and at the same time neglecting all aspects in which these successive mental operations *differ from each other* (e. g. their point in time, the difference of sensory material, and so on), the child is thus able to form the concept of apples. Only at this higher level of relating or coordinating several mental operations, the genesis of the concept of apples becomes explainable. The insight that the *sameness* lies at the level of coordinations also clears up why it is possible to integrate two or more spatially separated objects into a whole that in turn can then be called ›apples‹. What we are facing here is not the integration of concrete parts into a concrete whole (e. g. body parts/body, cards/deck of cards or countries/continent), neither it is the integration of concrete instances

[2] Piaget subdivided these categories of abstraction further so that he timewise distinguished between four kinds of abstraction (cf. Piaget, 2014, p. 317-323; see also Glasersfeld, 2003, p. 103-105). However, for the intentions of this article, the major distinction between empirical and reflecting abstraction will suffice.

into an abstract universal (e.g. concrete apple experiences/abstract concept of apple), but, instead, it is all about *an integration of several successive acts of integration*. This is, what makes the concept independent from immediate sensual material. So far, so good, but what, one could object, do apples have to do with mathematics? To cut it short: ›A lot!‹. In Piaget's words: »It seems to me that this [reflecting abstraction] is the basis of logical and mathematical abstraction« (Piaget, 1971, p. 16).

Mathematical abstractions: the case of Number

It is Frege's merit to have lucidly shown that numbers, similar to the concept of ›apples‹ in particular and the concept of plurality or multiplicity in general, cannot be abstracted from our concrete sensory experience. We cannot conceive of numbers as attributes or properties of certain complexes of our sensory experience because one and the same sensory impression always gives rise to a majority of different numbers: »While looking at one the same external phenomenon, I can say with equal truth both ›It is a group of trees‹ and ›It is five trees‹, or both ›Here are four companies‹ and ›Here are 500 men‹. Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another. This suggests [...] that the content of a statement of number is an assertation about a concept« (Frege, 1970, §46). Whether a certain sensory impression gives rise to this number or another, hence solely depends on the way in which we *conceptually* organize or structure this experience. As a consequence, the processes of abstraction by means of which all knowledge about numbers is constructed can most likely be modeled as reflecting rather than as empirical abstractions. But what exactly are the mental operations which form the material for these reflecting abstractions? One quite wide-ranging answer to this question was given by Hans Freudenthal: »In the genesis of the number concept«, he writes, »the counting number plays the first and most pregnant role« (Freudenthal, 1973, p. 191). Consequently, reflecting abstractions are exercised on *acts of counting*.[3] The child needs to become aware of the effects that a particular transformation exerts on the counting number and then learn little by little to separate those transformations on the counting procedure, which *change* the counting number, from those transformations that leave it *unchanged*. In the language of modern mathematics, one might also say that the child must learn a bunch of *invariances* of the counting number. Following the German sociologist Niklas Luhmann, we can seek for transformations and invariances in three ›meaning dimensions‹ (cf. Luhmann, 1995, p. 59-102):

1) the *factual* dimension (e. g. a substitution of one conceptual structuring by another one that allows a one-to-one mapping; or: spatial transformations such as counting from left to right or from right to left),

2) the *temporal* dimension (e.g. counting today, tomorrow, and next week),

3) the *social* dimension (e. g. me, my brother, and my father counting).

To become aware of all these invariances (and there are many more), the child must *relate* several acts of counting with each other and *analyze* these acts into *sameness* and *difference* of the counting results. That is, the child must perform reflecting abstractions in multiple dimensions to find out under which transformations the counting number is invariant and under which it is not.

[3] If we start our analysis by presupposing the ability of counting, then we jump right into the middle of the psychological genesis of the number concept. A comprehensive analysis would have to begin way earlier, but this lies beyond the scope of this paper (see cf. Glasersfeld, 2003, p. 160-175; Husserl, 2003, p. 9-90).

Concluding Remarks

To conclude, I want to generalize an important insight advocated by Freudenthal (1993, S. 192): All the above-mentioned invariances are crucial in the *psychological* genesis of the number concept; however, beside the relation of one-to-one correspondence between different conceptual segmentations, none of the above-mentioned invariances of the counting number can actually be formulated *within* mathematics. Having Luhmann's meaning dimension in mind, this leads to a very general question that might be a fruitful starting point to *re-think the history of mathematics*: Could it be a general phenomenon within the history of mathematics that the social genesis of mathematical concepts can be reconstructed as a systematic attempt to eliminate all those aspects of a concept that either lie in the temporal or social dimension?

References

Bruner, J. S., Goodnow, J. J. & Austin, G. A. (1986). *A study of thinking*. London: Transaction publishers.

Frege, G. (1970). *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number*. New York: Harper & Brothers.

Freudenthal, H. (1973). *Mathematics as a educational task*. Dodrecht: Kluwer.

Von Glasersfeld, E. (2003). *Radical constructivism*. London: Routledge.

Husserl, E. (2003). *Philosophy of Arithmetic*. Dodrecht: Springer.

Locke, J. (1836). *An essay concerning human understanding*. London: T. Tegg and Son.

Luhmann, N. (1995). *Social systems*. Stanford: Stanford University Press.

Piaget, J. (1971). *Genetic Epistemology* New York: W.W. Norton & Company Inc.

Piaget, J. (2014). *Studies in reflecting abstraction*. New York: Psychology Press.