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"MATHEMATICS AND LIVING TOGETHER": WHY, HOW, WHAT?

Power-relations in participatory action research projects in mathematics education

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Abstract. In this paper we describe a study on how mathematics teachers and researchers in action research projects experienced power relations. The power relations were produced between different actors within the projects, but also between the project participants and elements in different decision levels, in the broader context. We present a model for participatory action research, and we give voice to experiences of teachers and researchers in relation to this model.

Résumé. Dans cet article, nous décrivons une étude sur la manière dont les enseignants et les chercheurs en mathématiques dans les projets de recherche-action ont expérimenté des relations de pouvoir. Les relations de pouvoir ont été produites entre différents acteurs au sein des projets, mais aussi entre les participants au projet et les éléments de différents niveaux de décision, dans un contexte plus large. Nous présentons un modèle de recherche-action participative, et nous donnons la parole aux expériences des enseignants et des chercheurs par rapport à ce modèle.

Introduction

The purpose of the study presented in this paper was to examine characteristics of power relations in action research projects in mathematics education. Our objective was to study power relations between researchers and teachers, and also vis à vis the institutional framing of the project. We position the study within a critical and social approach and pay attention to inequities that may concern power relations between researchers, mathematics teachers and other actors (Skovsmose & Borba, 2004). We draw on Gellert (2008) who argues that if "professional development of mathematics teachers is considered a collective affair, then the concepts used to describe the teachers' actions and cognitions should reflect this perspective" (p. 94). We also draw on Atweh (2004) when we problematise "the process of research itself and critiques it in terms of power relationships between the participants" (p. 194). Atweh characterise participatory action research (PAR) as:

- a *social activity*. This research always recognizes the broader institutional context as part of the process, for example the municipalities or colleagues etc.
- a *participatory* action research. This research engages teachers and researchers to investigate their own knowledge and actions. A consequence is that people can only carry out action research 'on' themselves. It is not research done 'on' others.
- research which involves *collaboration* where teachers and researchers engage in research together. Everyone in the project strives towards developing her/his own professional competence with the support of each other.

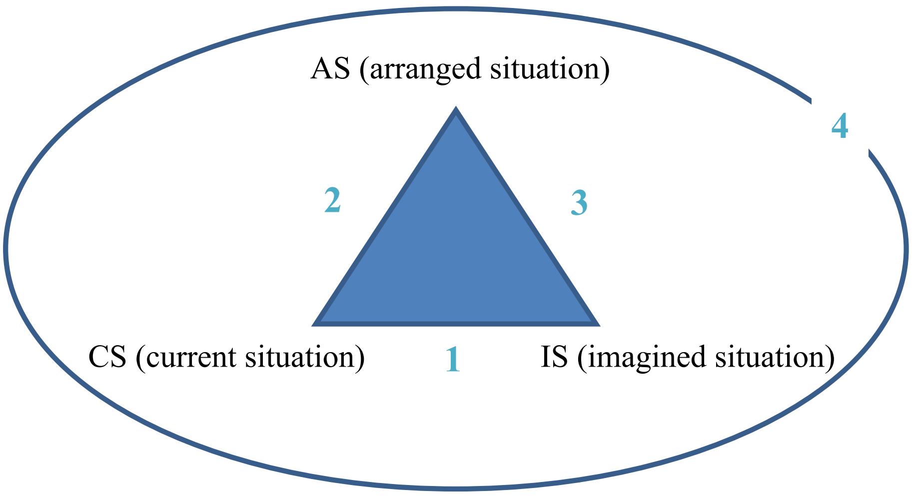
- research which is *emancipatory* and *critical*. Participatory action research may afford teachers to see and analyse the mechanisms that put limitations to their work as mathematics teachers. Included here is that the research itself seeks to challenge a mathematics teaching that do not provide equal opportunities for all students to learn mathematics.
- research which is *reflexive* in that it goes in two directions. The participants in the research investigate their practice, *and* they also aim to change it.

The points above were used as an analytical framework in this paper.

Data production and analysis

The data for this paper derive from three finalised action research projects. In each of them, four mathematics teachers participated from one school each. Two researchers were also participants in these projects. All projects were carried out in classrooms in the Swedish compulsory school, with the aim to develop practices in mathematics classroom. The three projects were all focused on student agency, assessment in day-to-day classroom work, and quality of mathematical discourse.

In the projects, a model by Skovsmose and Borba was used as a framework for the research processes (2004, see also Boistrup and Norén, 2013). The model is shown below.



In the model, CS refers to the *current situation* in the mathematics classroom before any substantial changes are introduced. IS corresponds to a vision about possible alternatives, an *imagined situation*, where the learning environment for the students is different. The third corner of the model illustrates the *arranged situation*. This situation is different from the current situation but also from the imagined situation. One could say that the arranged situation is "a practical alternative which emerges from a negotiation involving the researchers and teachers, and possibly also students, parents, and administrators" (Skovsmose & Borba, 2004, p. 214).

The numbers 1, 2, 3, and 4, refer to processes during the project, related to the corners of the model. They are here presented in the light of the projects of this paper:

1. Pedagogical imagination. In the process of pedagogical imagination, teachers (together with the researchers) discussed how they wanted to develop their teaching practice towards the "imagined situation", as well as investigating this process. Several ideas came up and the research group (consisting of teachers and researchers) had to select one aim of research that they agreed on.

2. Practical organisation. Here the teachers did their best to make changes in their mathematics classrooms in line with the respective project's aim. These practical organisations were discussed at the seminars with researchers and teachers, often outgoing from filmed sequences from the participating mathematics classrooms. The outcome is the "arranged situation".

3. Explorative reasoning. This process provides a means to draw conclusions not only in relation to the arranged situation but also in relation to the imagined situation. When we came to this process we summarized the project with help from participant logs and notes. Teachers and researchers learnt about assessment in mathematics classrooms through analysis of the arranged situation.

4. Scrutinizing the institutional context. In this fourth process, the teachers and researchers jointly analysed the institutional context and how it affected classroom communication and assessment in mathematics. While the situation in the classroom was in focus in the explorative reasoning, the institutional context was in focus in the fourth process.

Data was collected for aim and research questions for respective project, but also for the action research process itself. The latter is the data analysed in this paper. One part of the data consists of participant logs. All participants wrote logs, where reflections of the action research process were included. A second part of the data was notes from meetings in the research group. Participants took turns in taking notes from these meetings. The third part of the data was produced after the three action research projects when participants wrote anonymous reflections about the power relations in the projects.

In the analytical process we went through the written data. For our research objective, we then searched for the carachteristics of participatory action research (PAR), as summarised by Atweh (2004) (see the introduction of this paper). Our aim was to identify if, and how, the participants gave voice to the points in the list. In the next section, we give examples of this analysis.

Example of analysis

Here we present excerpts from the last set of data, collected after the projects' completion. The examples given are an illustration of a more complete data collection. When we collected the data, we really aimed for getting also critical or problematising reflections from the teachers. In the data collection situation reflected in this paper, all participants, 12 teachers and 2 researchers wrote anonymous accounts about the action research processes. All participants were asked to both describe notions that were experienced as rewarding *and* notions that were more problematic. It was not possible for the researchers to identify the identity of the teachers in their writing.

The first three excerpts are written by teachers and concern mainly characteristics of power relations between researchers and teachers. The fourth excerpt, also written by a teacher, represents mainly characteristics of power relations vis à vi the institutional framing of the project. The fifth excerpt is written by one of the researchers. We explain the analysis of each excerpt. For the presentation of the analysis, we adopt wordings from the list describing participatory action research (see the introduction). The excerpts are connected to the processes 1-4, presented above.

The first excerpt is connected to the first process of the model above, pedagogical imagination. Inger (teacher) reflects on power relations for the part of the project when the participants formulated the aim for the action research project:

Strangely enough it always felt like we were on the same level and that we all gave and took equally. I thought I would feel like "What can I offer a 'pro' like a researcher?" All in the group collaborated –it was a give and take. Ideas were flowing and all were running the project together. I(!) was important with my knowing as a mathematics teacher!

In the analysis we identified that this utterance reflected participatory action research (PAR) as a *participatory* activity. Inger explains that she as a teacher was fully included in the research team and that "all were running the project together". Furthermore, we could identify that the utterance reflected PAR as a *collaboration* ("All in the group collaborated").

The second excerpt is connected to the second process, on practical organisation. Elin reflects on how the work on developing the teaching and learning of mathematics in her classroom was facilitated through communication within the research group:

As I said earlier, this [cooperation] was made easy since we were filming and this way one could be involved also in the colleagues' classrooms. We took one person at a time when we analysed and this way everybody was involved and we could help each other in a cooperation in our role as [mathematics] teachers.

From the excerpt above we could identify PAR as involving *collaboration*. Elin described how the collaboration was facilitated through the filming and how this made it possible to 'visit' each other's mathematics classrooms. We could also identify PAR as *reflexive*, meaning that the research goes in two directions. It was clear to us from the excerpt, that Elin and the other teachers in the project wanted to change the teaching practice in mathematics. It was also clear that Elin describes the analytical process that simultaneously took place.

The following excerpt represents the third process, on explorative reasoning. Daniella wrote

It was a very rewarding cooperation which helped me to a deepened knowledge within mathematics, and made me to change my teaching. It was really positive to have the opportunity to have time for discussions and reflections. Moreover, to have researchers (Lisa and Joakim) who put words on this made my development to go one step further than if I only had discussed with colleagues.

This excerpt was analysed to reflect the research as *participatory*. Daniella focuses on her own development, and included here is that she, with the help of other teachers and the two researchers, could perform analysis of her own work, i.e. research 'on' herself. We could also identify traces of PAR as *emancipatory* and *critical* from this excerpt. Daniella expresses that new words, introduced by the researchers, helped her in her development. This was interpreted as Daniella beginning to see some mechanisms that limited her work, and she also could start develop her teaching.

Alice's account, in the excerpt below, is connected to the fourth process of scrutinizing the institutional context. In this excerpt, the institutional context is represented by the steering documents:

The feeling of getting the opportunity to look at both steering documents and oneself etc. critically was really rewarding. That strengthened me as a teacher.

Our analysis revealed that this account reflects PAR as a *social* activity. Alice is recognizing the broader institutional context when she reflects on her experiences from analyzing the steering documents. In this case this concerned the national syllabus in mathematics. We also read from the excerpt that this way to proceed was *emancipating*, since Alice described that it was rewarding, and strengthened her as a teacher.

On a whole, teachers and researchers gave account for processes which were characterised by positive reflections of the processes included in the action research project. One of the researchers took a more critical position towards her own actions:

With the model (the triangle) and process 3, and 4, this [action research as a social activity] was clearly present. I tried to problematize power, but I still think that next time, one of the researchers will have this as her/his special responsibility. Still, the other researcher took this responsibility sometimes. I think acting in a more self-reflective way developed during the project so that I worked more and more consciously with power issues.

The excerpt above reflects the possibility for the researcher to develop when moving on to a subsequent project. This was analysed as an example of how the PAR was *participatory* also for the researchers, meaning that also the researchers had the possibility to perform research on themselves.

Summary of findings

In the following we summarise our findings drawing on the list by Atweh (2004) presented above.

Participatory action research (PAR) as a social activity: In the project we strived towards valuing differences in participants' experiences and to make everyone's voices important. Our analysis of data indicates that this is in line with how it was perceived by the participants. Teachers went from a role as an observing participant to a participatory researcher, while the responsible researcher went from research leader to a facilitating reflexive participatory researcher. To some extent the institutional framing such as decisions made outside the classroom were addressed and challenged.

*PAR as participatory research***:** The teachers gave in the data account for how valuable it was to perform research on their own practise and how they felt included in the process on equal terms. From the researchers' logs, the analysis indicates that they learned a lot from the teachers and developed in their own roles as researchers.

*PAR involves collaboration***:** Very common in data are accounts of how the participants valued the collaboration with peers as well as with researchers or teachers. The discussions and reflections are mentioned as important.

*PAR as emancipatory and critical research***:** A common theme here is that the teachers felt emancipated to act against a "traditional" mathematics teaching practice with a completion of all of the items in a text book as a main goal of the mathematics teaching and learning. Critical aspects that were mentioned were to discuss the national syllabus in mathematics critically or to work more deep with each topic in mathematics and for a longer time. The researchers stated that teachers formulated other kinds of questions than in common research in mathematics teaching. These questions had critical relevance for practice as well for research.

PAR is reflective research: Both teachers and researchers describe how they changed their own practices. During the project it was the teachers practice that was investigated and the study of this paper is part of an investigation of the researchers' practice as well.

Discussion

This paper, very much tells a story of successful action research projects, at least in terms of power relations between participants. We argue that this is not due to an over interpretation of the data. As described earlier, the teachers were fully anonymous, and we invited all participants to write critical or problematising reflections if appropriate. Despite this, all accounts from the teachers resonated in positive ways about the processes of working in such a research project. One reason for this may be possible to be found in the institutional context of these project. They were all carried out in the same municipality and the projects received a lot of support during the course of the actions. Our experiences from other action research projects are that this is not always the case. Our assumption is, that when the municipality is not equally supportive, we would be able to identify also more 'negative' reflections.

This paper is focussing on the action research processes per se. This means that the mathematical content of the projects is not present in this particular paper, even though all three projects had a strong mathematical focus. We contend, however, that it is essential to address and discuss ways to perform research *together with* mathematics teachers, instead of *on*. There is a very dominating discourse, and has been since long, in Sweden when it comes to mathematics education. This tradition can be characterised as a strong presence of the discourse of "Do it quick and do it right" instead of a discourse where "Reasoning takes time" occurs regularly (cf Boistrup, 2017). Our assumption is that in order for mathematics education research to actually challenge such a discourse, research together with mathematics teachers is needed, while also paying attention to power relations. Through such a research, the actual context of mathematics teachers, is part of the research, and the research questions asked are more likely to concern actual problems in the teaching and learning of mathematics. This way, and as Gellert (2008) writes, professional development of mathematics teachers may become a collective affair, including the teachers.

References

Atweh, B. (2004). Understanding for changing and changing for understanding. Praxis between practice and theory through action research in mathematics education. I P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (sid. 187-206). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Boistrup, L. B. (2017). Assessment in mathematics education: A gatekeeping dispositive. In H. Straehler-Pohl, N. Bohlmann & A. Pais (Eds.), *The disorder of mathematics education. Challenging the sociopolitical dimensions of research* (pp. 209-230). Cham: Springer.

Boistrup, L B., & Norén, E. (2013). Power relations in mathematics education: Researching assessment discourses in day-to-day communication in mathematics classrooms. In M. Berger, K. Brodie, V. Frith and K de Roux (Eds.). *Proceedings from Mathematics Education and Society 7th International Conference*.

Gellert, U. (2008). Routines and collective orientations in mathematics teachers' professional development. *Educational Studies in Mathematics, 67*, 93–110.

Skovsmose, O., & Borba, M. (2004). Research methodology and critical mathematics education. I P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (pp. 207-226). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Does new educational technology compensate my ignorance?

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Abstract: Technology as seen by casual folks is a miracle that compensates any human's weaknesses and they even imagine of replacing teacher by machine! This view would be true if it does not intervene with teaching field and if the word 'any' is replaced by 'some': physical weakness can be compensated for, or at least balanced by technology but ignorance would be piled up by mythical believe in technology. Examples that investigated here show the insufficiency that when conveyed by ignorance leads to implausible answers and results.

Keywords: educational technology; teaching; learning; GeoGebra; MATLAB.

Introduction

Technology as defined in Merriam-Webster's dictionary is "the specialized aspects of a particular field of endeavor, for example educational technology" [1]. And educational technologies as discussed by Larry Cuban [2] range from textbooks, black(white)board, over head projectors (old technologies) to computers, software, and digital cameras and so on (new technologies). So the soil tablets used for practicing writing and reading in our village about nine decades ago is a sort of educational technology. Those tablets were blind user obeying that did not intervene the teaching and learning process. But using technology like TV can be harmful and caused retarding in language learning in early years of kids life []. And modern technology like MS-Word can offer spelling and grammar corrections that may affect the ignorant learner/teacher negatively.
A special spherical lens and paper folding an ellipse described below//here, are examples that show how technology can facilitate conducting cumbersome or time consuming//demanding experiments easily.

A Special Spherical Lens

A bundle of parallel light beams strike a clear sphere (Figure 1). What would be the refractive index, if all of the light beams emerge from the point A on the opposite side of the sphere?

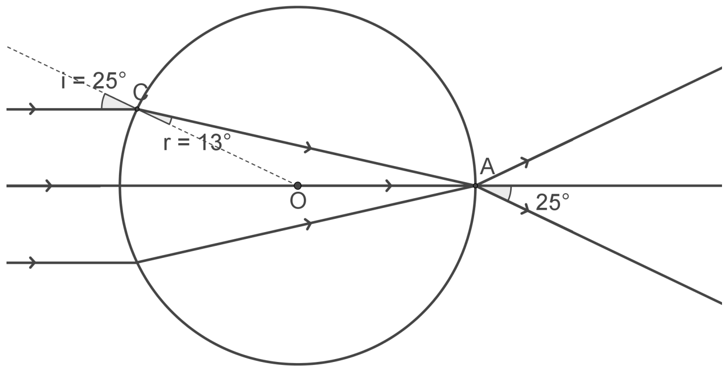

Figure 1- A special spherical lens.

The angles r and i are independent of radius of the sphere and they determine the index of refraction and thence//subsequently the type of material that can refracts the light beams in desired direction.

For any medium with certain refractive index there is mathematically one ring on the sphere that directs the light beams towards point A. So, many numbers of refractive indices and light wavelengths are needed to conduct the experiment i.e. for a certain wavelength, different media needed and vice versa. This demanding experiment can be easily conducted by computer simulation. Figure 1 shows a snapshot of experiment done by GeoGebra.

Paper Folding an Ellipse

Mark a point on a circular disk of paper then fold the disk so that its edge touches the marked point. Ink the fold line in. If you repeat the process for at least forty times then an ellipse would be seen surrounded by the fold lines [3]. Figure 2 shows a snapshot of simulated activity.

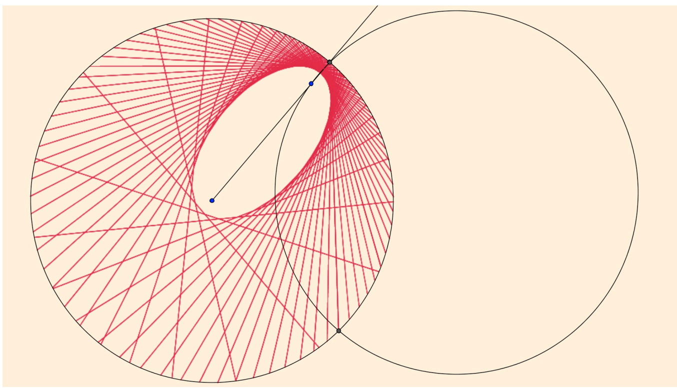

Figure 2- Paper folding an ellipse simulated by GeoGebra.

When the Machine Gives Wrong Answers

The following examples shows how the technology demands more knowledge and experiences to give correct results in teaching and learning mathematics.

Plot $\lfloor sin(x) \rfloor$

The curve of $\lfloor sin(x) \rfloor$ as graphed by GeoGebra and the ezplot command of MATLAB is depicted in Figure 3. A high school student can easily seize the error in the graph; maximum of $sin(x)$ is one not zero. Thus there must be a point with y-coordinate equals one (y = 1). The point would be appeared if a fine step size chosen, using slider as x-coordinate and tracing the curve with the point (t , floor(sin(t))) where t is the slider value.

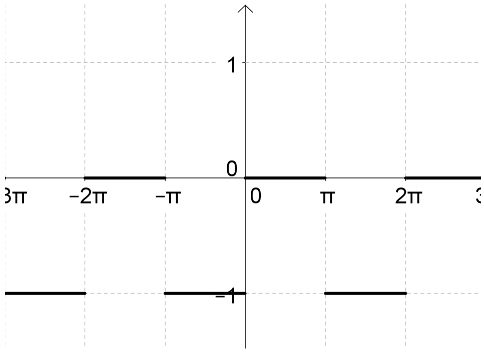

Figure 3- Graph of floor(sin(x)).

The same method can be used for any $\lfloor f(x) \rfloor$ with extrema that graphing software jump//spring over them because of unadjusted step size.

Plot $a + \sqrt{\ln(\sin(2\pi t x))}$ for real values of x

Typing this function into the input bar of GeoGebra return nothing and MATLAB, using ezplot or even plot(x,y) command, shows a plot of imaginary parts against the real parts. As Pablo Picasso said "computers are useless, they can only give you answers." And the answers may be incorrect. Let check the function to help the computer plot it. As the plot of $\sin(2\pi t x)$ for t = 0.5 in Figure 4 (red dashed curve) shows, for real values only the upper half wave of sinusoidal function can be feed into natural logarithm *ln()*. And the last function, the blue solid curve, can in turn feed the square root function only by zeros. So the desired graph is a series of points with $y = a$, see Figure 5 for $a = 1$.

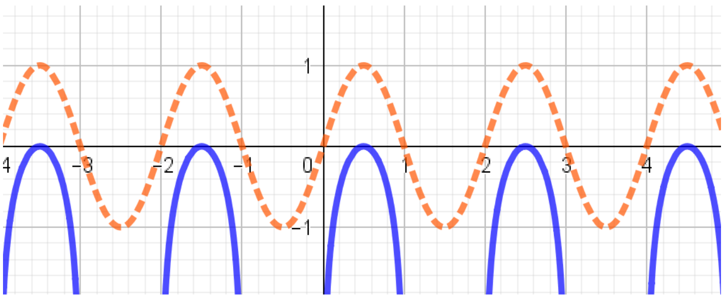

Figure 4- The red dashed curve is sin(2πtx) and the blue solid one is ln(sin(2πtx)) .

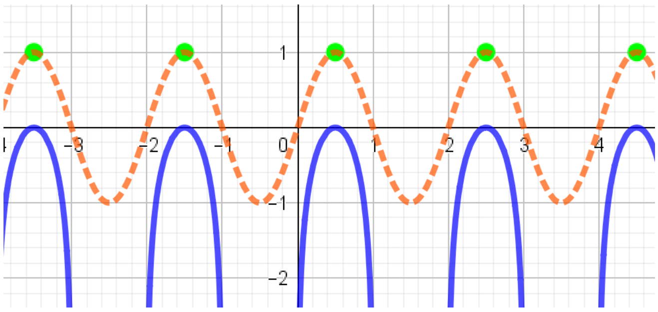

Figure 5- The green points are the plot of $1+\sqrt{\ln(\sin(2\pi x))}$.

Drawing a Series of Inscribed Squares

Table 1, at the end of this paper, shows two GeoGebra constructions for drawing a series of inscribed squares. Method 1 works only for drawing square by joining the midpoint sides of preceding one. Moving away from the midpoints is not easy and clear. However it would be very easy//easily done by the same routine just by applying complex variables – method 2 [4].

Palindromic Numbers

The reversed number of 123 is 321 and if the reversed number and its origin read the same then that number is Palindromic. For example 454 is Palindromic. A Palindromic number can be produced from a non Palindromic one by the following procedures which explained by an example:

The origin number a = 153 and its reverse is b = 351, the addition of a and b is 153 + 351 = 504 which is not Palindromic so we add it to its reverse to obtain 909 which is Palindromic. We may continue reversing and adding for any number of times until we obtain a Palindromic number. The following script for MATLAB creates a Palindromic number and counts the reversing-adding steps. Working with numeric values avoided because of limited precision of machine.

Can this iterating scheme generate palindromic number from any number? Can technology decide//determine this problem? Without an insightful and experienced brain the answer is no[].

```
%%%%%%%%%%%%%%%% symet.m %%%%%%%%%%%%%%%
%       Try 978 as an input but take care of your computer - it is a harsh
%       number. How to better the algorithm to obtain a Palindromic
%       number from a harsh number like 978 or 1997?
%==============================================%

clc;
A = input('Input the non-symetric number    ');
carry1into = 0; gauge = 0; counter = 0;
A = num2str(A);
L = length(A);
B = fliplr(A);
while strcmp(A,B) ~= 1
```

```
  for k = 1:L
    c = str2num(A(L-k+1)) + str2num(B(L-k+1)) + carry1into;
    if c<=9
      gauge = c;
      carry1into = 0;
    else
      gauge = c - 10;
      carry1into = 1;
    end % if
    B(L-k+1) = num2str(gauge);
    if k == L & carry1into == 1
      B = ['1' B];
    end % if
  end % for
  A = fliplr(B);
  counter = counter + 1;
  L = length(A);
  carry1into = 0;
end %while
fprintf(1, '\n\nThe symetric number is:\n\n',counter)
disp(A)
fprintf(1, '\n\n%25f Reversing-adding process needed.\n',counter)
```

Conclusion

The conclusion is to rephrase the proverbs "a teacher is better than two books" to 'a conscious teacher/student is better than a computer': Educational technology not only cannot compensate ignorance in teaching and learning but also increases the demand for more knowledge and experiences. In other hand, the fault in the educational technology can be compensated by human's comprehension and consciousness. Even these//those bugs can be switched//turned into teaching and learning opportunity by calling more facts and skills from the related topic//field.

References

1 – Alice Park Monday, Aug. 06, 2007, Baby Einsteins: Not So Smart After All. Retrieved on 2018-01-15 from: http://content.time.com/time/health/article/0,8599,1650352,00.html
2 – Larry Cuban (2001). Oversold and underused computers in the classroom (page 12). Harvard University Press.
3 – Brian Bolt (1982), Mathematical Activities: A Resource Book For Teachers (page 10), Cambridge University Press.
4 – Ted Courant, GeoGebra Materials Retrieved on 2018-02-13 form: https://www.geogebra.org/m/AauBspSS,

No	Name	Description	Value	No	Name	Description	Value
1	Number a		a = 8	1	Complex Number z_1		z_1 = 1 + ί
2	Point A		A = (-4, -2)	2	Complex Number z_2		z_2 = 1 - ί
3	Point B	Point on Circle(A, a)	B = (4, -2)	3	Number Ratio		Ratio = 0.02
4	Polygon poly1	Polygon(A, B, 4)	poly1 = 64	4	Complex Number zz	(1 + ί) / (Ratio + ί)	zz = 1.0196 - 0.9796ί
4	Segment g	Segment A, B	g = 8	5	Number n		n = 1
4	Segment h	Segment B, C	h = 8	6	List list1	Sequence(z_1 / zz^k, k, 1, n)	list1 = {0.02 + 1ί}
4	Point C	Polygon(A, B, 4)	C = (4, 6)	7	List list2	Sequence(z_2 / zz^k, k, 1, n)	list2 = {1 - 0.02ί}
4	Point D	Polygon(A, B, 4)	D = (-4, 6)	8	List list3	Sequence(Polygon(Element(list2, k), Element(list1, k), 4), k, 1, n)	list3 = {2.0008}
4	Segment i	Segment C, D	i = 8	9	Polygon poly1	Polygon(z_2, z_1, 4)	poly1 = 4
4	Segment j	Segment D, A	j = 8	9	Segment a	Segment z_2, z_1	a = 2
5	Number b		b = 1	9	Segment b	Segment z_1, A	b = 2
6	Point E	Midpoint of A, C	E = (0, 2)	9	Complex Number A	Polygon(z_2, z_1, 4)	A = -1 + 1ί
7	List L_1	Sequence(Polygon(Dilate(Rotate(A, (k π) / 4, E), (sqrt(2) / 2)^k, E), Dilate(Rotate(B, (k π) / 4, E), (sqrt(2) / 2)^k, E), 4), k, 1, b)	L_1 = {32}	9	Complex Number B	Polygon(z_2, z_1, 4)	B = -1 - 1ί
				9	Segment c	Segment A, B	c = 2
				9	Segment d	Segment B, z_2	d = 2

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Co-constructing teaching and learning spaces in and between mathematics and physics at school

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Abstract. In this paper, we view the school unit as an open learning organisation. We focus on the appearances of mathematical signs common in school mathematics and physics to investigate the variety of the associated implicit uni-disciplinary and interdisciplinary meanings; as school courses and as scientific disciplines. We draw on the teachers' reflections upon signs that appear in mathematics and physics textbooks, in order to reveal the diverse co-existing, often diverging, cognitive processes, intentionalities and conventions, implicit for both the learners and for the teachers of the different courses. We introduce a communicational semiotic system, which consists of a basic semiotic triad (sign, interpreter, epistemic object) to investigate the emerging communicational meanings. Our approach allows for the identification of invisible, interdisciplinary obstacles to the understanding of each discipline. This complex teaching-learning space renders possible the distinctions, the linkings and the constructions of uni-/interdisciplinary meanings.

Résumé. Dans cet article, nous considérons l'unité scolaire comme une organisation apprenante ouverte. Nous nous concentrons sur les apparences des signes mathématiques communs en mathématiques scolaires et en physique afin d'étudier la variété des significations implicites mono-disciplinaires et interdisciplinaires associées aux cours scolaires et aux disciplines scientifiques. Nous nous appuyons sur les réflexions des enseignants sur les signes qui apparaissent dans les manuels de mathématiques et de physique afin de révéler les divers processus cognitifs, les intentionnalités et les conventions coexistantes, souvent divergentes, implicites pour les apprenants et pour les enseignants des différents cours. Nous introduisons un système sémiotique communicationnel, qui consiste en une triade sémiotique de base (signe, interprète, objet épistémique), pour étudier les significations communicationnelles émergentes. Notre approche permet d'identifier un espace d'obstacles invisibles et de niveau interdisciplinaire à la compréhension de chaque discipline. Cet espace complexe rend possible l'observation des distinctions, des liens et des constructions de significations mono-/interdisciplinaires pendant l'enseignement et l'apprentissage des mathematiques et de la physique.

Unidisciplinary realisations of interdisciplinary realities

The official descriptions of the school unit objectives, structures, rules and social representations reflect a perspective of the school unit courses as a 'puzzle' of distinct parts. When placed together they form an assumed corpus of knowledge consisting of carefully cut parts that they fit each other towards the big picture of the assumed corpus, but they hardly interact with each other. Exactly like the pieces of a puzzle, the courses seem to lack an obvious meaning with respect to the big picture, until that picture is formed. The students are taught courses, which are named according to the established system of scientific disciplines, but they do not reflect the disciplines the name of which they bear. In contrast with the scientific reality of interacting and continuously evolving disciplines, the content of the courses is static and not linked with each other. In the current school reality, the students are left on their own to discern the fact that the different courses are pieces of the 'puzzle' and, importantly, the way they are linked. The students are expected to learn what is presented to each course in a 'leap of faith', waiting for the 'big picture' to be revealed, which will render each course meaningful to them. This artificially 'clear-cut' compartmentalisation of knowledge is usually linked with centralised educational systems, in which the central planification is more important than the local dynamics and their interactions.

In this study, we embrace the complexity of a 'real' school unit that functions as a complex, learning organisation (Davis & Simmt, 2003; Romme & Dillen, 1997). Through this perspective, the implicit and explicit relationships amongst roles, intentionalities, functions and structures (for example, a son may also be a teacher and a principal and a parent of a student who is at the same time a daughter and a sibling) are interwoven in an *interdisciplinary* reality, constituting a complex whole that transcends even the technologically redefined time-space boundaries of the school unit (Moutsios-Rentzos, Kalavasis & Sofos, 2017). Interdisciplinarity involves the dialectic and transforming linkings of those concepts and methods specific to each discipline and those common to all (Piaget, 1974).

In line with this interdisciplinary, systemic perspective of the school unit, learning may be conceptualised as *linking links* towards the transformation of the system in a qualitatively novel functional equilibrium (Moutsios-Rentzos & Kalavasis, 2016). We posit that such learning occurs through the continuous communications (Watzlawick, Beavin & Jackson, 1967) of the protagonists' intentional reflections upon actions and experiences.

In this paper, we discuss the first phase of a broader project which is built upon these ideas. The project concerns the *appearances* of *common mathematical symbolism (signs)* in the school mathematics and physics, with the purpose to *engineer* appropriate a communication space for the teachers of the two courses. At the crux of this approach lie a series of *co-laboratories* with the teachers' experiencing and intentionally reflecting (Jay & Johnson, 2002; Nissilä, 2005) upon their intersubjective experience of those appearances to be actively engaged with the design of teaching activities that draw upon those experiences and reflections. Such an approach acknowledges and builds upon the fact that the students do encounter the same mathematical symbolism in the different courses, positing that the communication between the different teachers is crucial for the students to give to their experiences of the phenomena distinct, yet connected, uni-disciplinary and interdisciplinary meanings.

Signs, interpretations and interpreters: a semiotic system

Central to our approach, is our perspective about the phenomenology of signs in an interdisciplinary teaching-learning approach. Several approaches to semiotics have been suggested both general (de Saussure, 1959; Peirce, 1981) and specifically with respect to didactics of mathematics (see, for example, the special issue 1-2 of Educational Studies in Mathematics, 2006). Our perspective draws upon a systemic perspective of communication (Watzlawick et al., 1967) and upon commonly cited in the literature *communicational triads*. Ongstad (2006, p. 256) clarifies that "by grouping the triads explicitly as mutual relations, one will get a systemic understanding that will alter the logic. By systemic is meant the dynamic and partly unpredictable character of a relational system as different from a closed, systematic stable system".

Following these, we introduce a *communicational semiotic system* (in the sense of Ongstad, 2006), which consists of a basic *communicational triad*: *epistemic object-interpreter-sign* (see Figure 1, left). The triad is activated by the *sign* includes all the means of signification that may be employed within a communication (including symbols, figures, kinesics, facial expressions etc). The *interpreters* are the roles that the individuals assume in the specific communicational processes. Hence, the same individual may constitute a different interpreter for in a communicational situation. For example, a teacher who is also the principal of a school unit may experience the communication situation in diverse ways depending on the role he/she assumes. The *epistemic objects* refer to the normative construction of an institutionally acceptable scientific object, as conceptualised by the interpreter in the given situation. Thus, the epistemic object in our triad is the super-sum of one's own epistemological conception of the signified object, as transformed by one's conceptions of what this object should be in the given situation. In our communication system, the communicational *meaning* emerges through the entirety of the communicational interactions, as a complex whole, rendering crucial the consideration of all the interacting elements of the communicational triad. Importantly, we do not assume the employed roles or the assigned objects, but we consider them as part of our investigating the emerging meanings.

By revealing to the interpreters, the common appearances of a sign to their courses, we give the opportunity to a relatively neutral communication space to emerge (in green colour, in Figure 1, right), which may serve as the common ground for the teachers of different disciplines to share their reflections and to collaborate. We assume this common space to be neutral, as it builds upon the textbook, which is external to the school unit reality and, crucially, the teachers are considered to be responsible for.

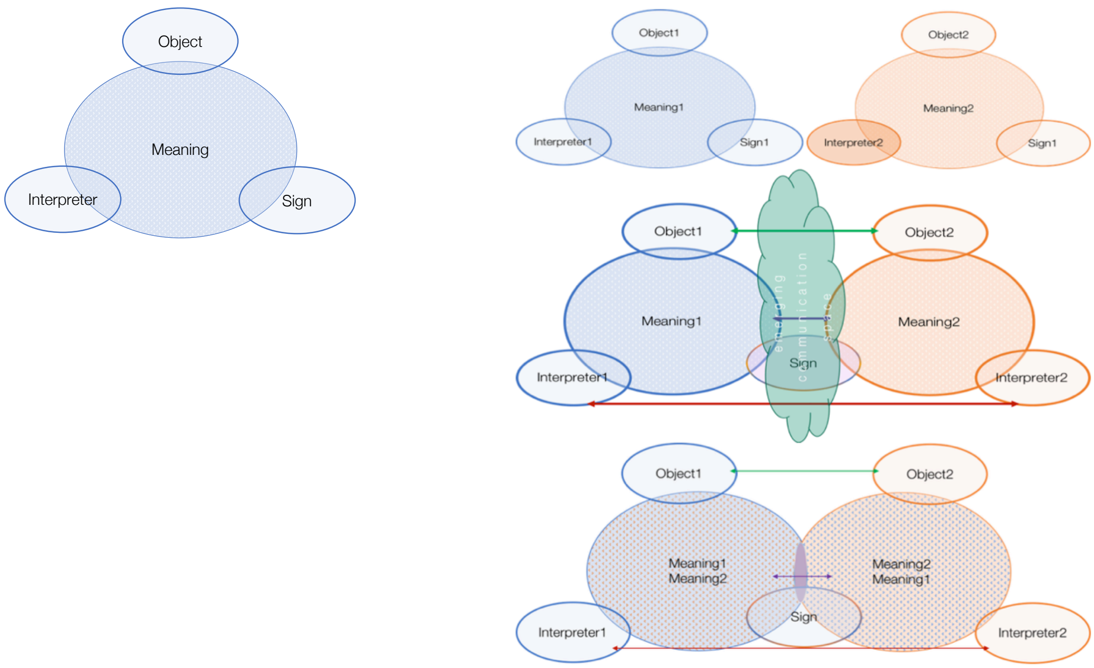


Figure 1. A schematic representation of the proposed perspective.

In Figure 2, the communicational triad is applied as a thought experiment about the sign "—" (the fraction line). We include some of the mathematical and physics objects that may be signified through the fraction line, as well as the potential appearances that the students may encounter in their school life. Though in Figure 2, the signs, the objects and the roles are real class examples, we *do not assume their relationships*, thus we do not assume the emerging meaning, rendering Figure 2 a representation of only one hypothetical communicational situation.

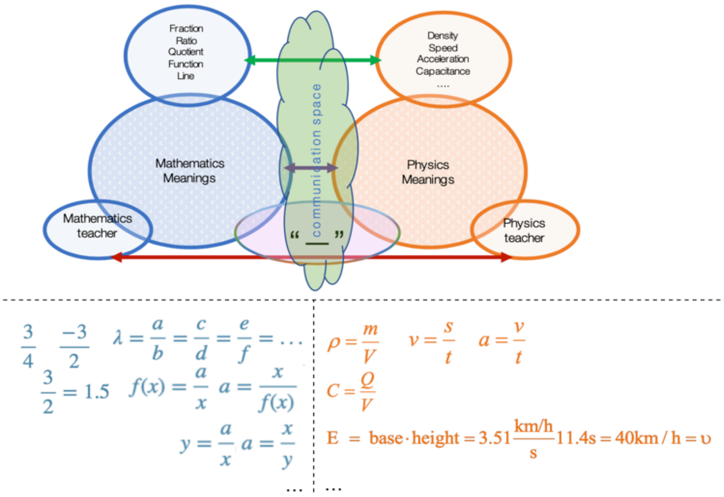
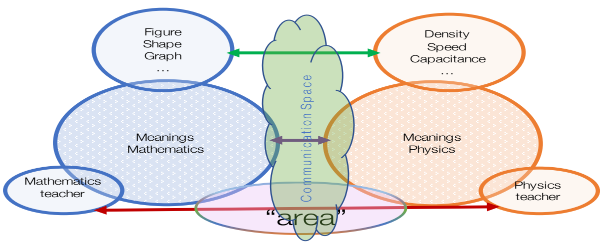


Figure 2. Applying the proposed perspective to the sign "—" (fraction line).

In Moutsios-Rentzos, Kalavasis and Kritikos (2017), we argued that this approach may help in revealing interdisciplinary obstacles with respect to unidisciplinary meanings. For example, we considered the definition of capacitance in the Greek Physics textbook, noting that the fraction line was linked with the mathematical notion of quotient of electric charge over electric potential, which posed an obstacle for the students to realise that the capacitance is a constant magnitude with respect to the electric charge and the electric potential (since, the fraction line is linked with the operation of division). Instead, we suggested that the definition may more appropriately link the fraction line with the mathematical notion of ratio, which in mathematics is commonly encountered as a constant: "Drawing upon the *mathematical* notion of ratio, the students may appropriately infer that the *physical* capacitance remains the same, implying that the charge held is also doubled, thus *mathematically facilitating* their gaining *deeper physical understanding*" (Moutsios-Rentzos et al., 2017, p. 298-299).

Interdisciplinary systemic reflections: the case of area

We built upon the aforementioned ideas to engineer a *co-laboratory* of a series of *reflective activities* for *teachers* of *high school physics and mathematics* of the same school unit, focussed on the *sign of "area"*, as appearing in the respective school textbooks (see Figure 3) and especially in the appearances (or not) of *area measurement units* (Moutsios-Rentzos, et al., 2017). In order to maximize the pragmatic effects of our planifications, we started from the *institutionally assigned sources of authority* within the school unit and in specific on the teachers.



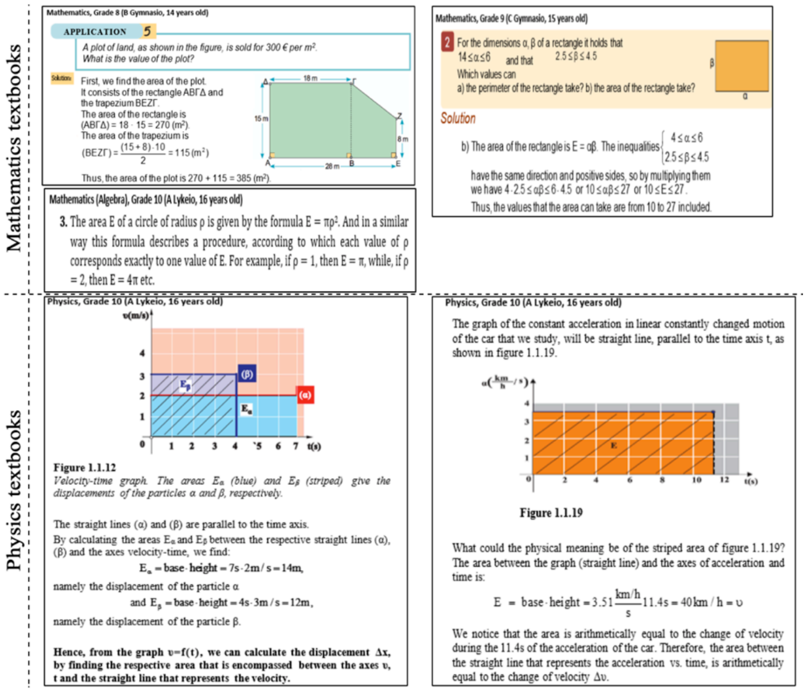


Figure 3. Appearances of the sign "area" in the Greek textbooks.

Through our allowing for the teachers to be exposed on the appearances of common signs in both mathematics and physics, we focus on making visible the seemingly converging assigned functions and meanings, usually considered to transcend disciplines. The appearances of the signs may act as the vehicle that will allow for the creation of a *reflectively functional* communication space within which the interpreters may commence to share their meanings and reflections. Implicit in our approach is the teachers' shared intentionality for the students to learn and the fact that, in Greece, the teachers of the different courses personify diverse scientific and professional internationalities (perspectives, motivations and expectations) and the real-world implications that these have (for example, the school hours allocated for each course, the professional teaching rights).

We conducted co-laboratories with 25 masters students (including, mathematicians and physicists, primary and kindergarten school teachers) and three experienced physics teachers. The mathematics teachers were surprised by the diversity of symbolism in the school textbooks: "Do these really exist in the school mathematics textbooks?" They were especially interested that measurement units did not appear in a consistent form or they were completely missing, while the expression "E=π" was especially problematic for them. The physicists consider the measurement units very important, as they focus on the algebraic form of area, which usually denotes another magnitude; the outcome of the area on speed-time graph is the distance (length). However, they seem not to realise that the notion of area is also a basic magnitude in Physics, measured in measurement units, which cannot be the measurement units of another magnitude.

The initial reluctance by the participants was followed by (sometimes passionate) discussions concerning epistemology, pedagogy, curriculum and (in)ability to implement change in school. Hence, we argue that the communication space was constructed, and we concentrate our efforts on the second phase of the project: implementation and incorporation of these planifications in everyday teaching, including more educational protagonists (for example, students and parents) of the school unit system.

In conclusion, we posit that through the proposed approach, the school unit complexity may be organised to allow for new qualities of learning both disciplines to be experienced and communicated, thus leading to a reflective dynamic circle of meta-learning, in which the meanings are interdisciplinary (with shared and distinct parts; see Figure 1), but the epistemic objects and the interpreters remain bounded to their distinct discipline: separate (not conflated) *and* connected. The disciplines retain their social and epistemic distinctness, through their continuous social and epistemic communication with different disciplines. Complexity may be impossible to be modelled in the traditional sense, but we argue that the proposed approach is a step towards reconciling 'ingenio', designo' and ethics (Le Moigne, 2013), crucially within the real-world constraints of the everyday reality of the school unit, thus being a step towards pragmatic interdisciplinary realisations of teaching and learning together, towards an interdisciplinary school reality within which the knowledge is collaboratively co-constructed by both the students *and* the teachers of mathematics *and* physics.

References

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137-167.
De Saussure, F. (1959). *Course in General Linguistics.* New York: McGraw-Hill.
Jay, J. K., & Johnson, K. L. (2002). Capturing complexity: a typology of reflective practice for teacher education. *Teaching and Teacher Education*, *18*, 73-85.
Le Moigne, J. L. (2013). The Intelligence of Complexity: Do the Ethical Aims of Research and Intervention in Education Not Lead Us to a New Discourse" On the Study Methods of our Time"?. *Complicity: An International Journal of Complexity and Education*, *10*(1/2), 1-17.
Moutsios-Rentzos, A., & Kalavasis, F. (2016). Systemic approaches to the complexity in mathematics education research. *International Journal for Mathematics in Education*, *7*, 97-119.
Moutsios-Rentzos, A., Kritikos, G., & Kalavasis, F. (2017). Functions of operations and operands in school mathematics and physics: a complex interdisciplinary (de)mathematised phenomenology. *Quaderni di Ricerca in Didattica (Mathematics), 27*(2), 297-299.
Moutsios-Rentzos, A., Kalavasis, F., & Sofos, E. (2017). Learning paths and teaching bridges: the emergent mathematics classroom within the open system of a globalised virtual social network. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and Technology. Advances in Mathematics Education* (pp. 371-393). Dordrecht: Springer.
Nissilä, S. P. (2005). Individual and collective reflection: How to meet the needs of development in teaching. *European Journal of Teacher Education*, *28*(2), 209-219.
Ongstad, S. (2006). Mathematics and mathematics education as triadic communication? A semiotic framework exemplified. *Educational studies in mathematics*, *61*(1), 247-277.
Peirce, C. S. (1981). *Philosophical writings of Peirce Selected and edited with an introduction by Justus Buchler*. New York: Dover.
Piaget (1974). L' épistémologie des relations interdisciplinaires, *Internationales Jahrbuch für interdisziplinäre Forschung*, *1*, 154-172.
Romme, G., & Dillen, R. (1997). Mapping the landscape of organizational learning. *European Management Journal*, *15*(1), 68-78.
Watzlawick, P., Beavin, J., & Jackson, D. D. (1967). *Pragmatics of Human Communication. A study of interactional patterns, pathologies, and paradoxes*. New York: Norton.

A theoretical framework to model the relationships between teachers and researchers in collaborative research

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Abstract: The collaborative work between researchers and teachers has been modeled by the Meta-Didactical Transposition (Arzarello et al. 2014). In this model, the relationships between the different communities lean on an internalization of external elements in the actors' praxeologies. At a deeper level of granularity, it is essential to consider the objects which are at stake in the common work of the two communities as well as the actions and activities that the two communities can perform on and with these objects. Starting from the concept of boundary objects, we present a framework allowing the analysis of the relationship between actors of a collaborative research.

Résumé: le travail collaboratif entre enseignants et chercheurs a été modéliser dans la Transposition Meta-Didactique (Arzarello et al. 2014). Dans ce modèle, les relations entre différentes communautés s'appuient sur un processus d'internalisation d'éléments externes présents dans les praxéologies des acteurs. A un niveau plus fin de granularité, il est essentiel de considérer les objets qui sont en jeu dans le travail en commun des deux communautés tout comme des actions et des activités qu'elles peuvent bâtir sur ces objets. En partant du concept d'objet-frontière, nous présentons un cadre permettant d'analyser les relations entre les acteurs d'une recherche collaborative.

Introduction

Several methodological trends in educational research rely on collaboration between teachers and researchers. These collaborative methodologies tend to work no longer "on" teachers, but "with" teachers. Within these different currents, such as action research (Lewin, 1946), Design experiment (Brown, 1992), didactic engineering (Chevallard, 1982 and Artigue 1992), Design based research (Design based research Collective, 2003, Swan, 2007) or design-oriented research, it exists an epistemological necessity for researchers to act with teachers. EducTice is a multidisciplinary team of the French Institute of Education in the École Normale Supérieure de Lyon. Its main research domain is the use of technology within education and most of its research are built on a methodology of collaborative research

(Design based research or didactic engineering). The issues of such research is widely shared within the team and a monthly seminar has been organized to reflect about the framework that underpins our research. In this text, we present the theoretical framework based on the Meta Didactical Transposition (Arzarello et al. 2014) and the concept of boundary object (Star & Griesemer, 1989) that we adapt to allow analysis of interactions between different communities. The link between MTD and boundary objects theory is both methodological and theoretical in the sense that boundary objects appear to be a driving force of MDT dynamics and that the analysis of this dynamic is based on an analysis of the boundary objects at stake in interactions between actors. We present in the first section the theoretical background of the Meta Didactical Transposition and the concept of boundary object, by highlighting the links between them. In the following section, we illustrate this methodological and theoretical construction by analyzing a dialogue between teachers and researchers.

Theoretical and methodological construction

The model of the Meta Didactical Transposition is born in Italy to better understand and analyze the relationships between teachers and researchers in the frame of a national teacher education program for in-service mathematics teachers supported by the Ministry of Education1. The MDT is the dynamic process through which both the didactical and the meta-didactical praxeologies of the different communities involved change within the institutional environment of the research. The term "meta-didactical" refers to the fact that a large part of shared knowledge are coming from reflection on the didactical process and that, as well as the concept of didactical transposition (Chevallard, 1999) considers the modification of mathematical knowledge from the place of creation of knowledge to the place of teaching, we consider the modification of didactical knowledge in the process of collaboration between communities.

An important feature of MDT is the concept of internalization: both communities start the common work with knowledge that, for some of them, are external, that is to say, are not available for action. Regarding the anthropological approach of Chevallard (Ibid.), we could say that the praxeologies are not the same in front of a given type of task, either in term of techniques or in term of justification of these techniques. The common work, the discussions, the observation and the analysis of class sessions allow to integrate into the actors' system of thought these external components. This phenomenon is called internalization (Arzarello & al., 2014) and has a pragmatic scope: because of the internalization, the component becomes a tool for action; the main hypothesis of the MTD is that, in a given institution, the internalization allows the actors to modify their praxeologies and participates to their professional development. Without entering the details of MDT, it is clear that the dynamic process must be initialized and then maintained in order to favor this internalization.

The first step of this dynamic lives in the possibility to speak together of a component seen as an object, taken in a sense of material or immaterial object, where the two communities have an interest to work with. We use here the concept of boundary object (Star & Griesemer 1989) taking into account the properties given to this boundary object when boundary is not seen as a line of demarcation but more as a boundary space. It is precisely within this boundary that the two communities are able to start the discussion. The first important characteristic proposed by Star & Griesemer (Ibid.) is called interpretive flexibility: each of the two communities is able to speak of this object and to act on it, even if, at the beginning, the

1 M@t.abel: https://www.youtube.com/watch?v=j8_naguqYLY

object itself, its potentialities and its properties may have different meaning for the communities. The example of formative assessment that will be developed later can give an interesting illustration: both teachers and researchers are able to speak of formative assessment, but the meaning that each of the actors gives to this term is not necessary the same. But, as important as it is, the flexibility is not the only characteristic of boundary object:

"The two other aspects of boundary objects, much more rarely cited or used, are (1) the material/organizational structure of different types of boundary objects and (2) the question of scale/granularity." (Star, 2010, p. 602)

Thus, the concept of object can be considered as a container, containing objects whose properties inherit from the properties of the container. The metaphor of object in the paradigm of object-oriented programming gives a good idea of the boundary object. In this paradigm, objects integrate both data and functions and are modeled after real-world objects , which the environment interact with. Encapsulation is a process of binding properties within a class seen as a container. Objects, within the class-container, inherit and extend properties, attributes and methods. It is often interesting to note that the actors do not speak directly of the boundary-object that is at stake in a collaborative work but rather of one or the other of its components. Furthermore, boundary objects exist if and only if someone acts on them. It is through the actions on the boundary object (or one or more of its components) that the boundary can evolve. The phenomenon of internalization will be precisely the extension for one or the other community to consider the properties highlighted by the action on boundary object in its own praxeology. Following Carlile (2004), we distinguish three types of activities that make possible a modification of the common boundary by enlarging the space of shared understanding of the object (container or component):

- the transfer activities concern the syntactic aspects of the boundary object necessary to be able to talk together about the object and to adopt a common vocabulary to designate this object,
- the translation activities affect the meaning of the objects (from a semantic point of view) on which the actors act,
- the transformation activities in which the actors share knowledge for the creation of new shared knowledge related to uses (from a pragmatic point of view).

Transfer and translation activities highlight the boundary and the common understanding of the object or one of its components while the transformation implements in the actors' praxeologies the result of the common work done on the object.

Thus, the analysis of actions on boundary objects and the understanding of the boundary space highlighted by collaborative work constitute the driving force behind the dynamics that will allow internalization and share of praxeologies. In the next section, we shortly present an analysis of a dialogue between researchers and teachers leaning on the boundary object analysis.

Exemplification

Looking at the interactions between teachers and researchers, we examine the different ways actors are speaking of the boundary object or some of the components of the boundary object. In term of methodology, the first step consists on spotting the components that the actors are speaking of, then highlighting the kind of action they are undertaking on these objects. So, it is possible to distinguish the syntactic action of transfer, the semantic approach of translation and the pragmatic approach of transformation. It is possible in these conditions to establish a link between the praxeologies of the actors and to follow the phenomenon of internalization. The small excerpt below that we analyze is taken from a broader data collection coming from the European Project FaSMEd2. Teachers and researchers are working together in order to design a formative assessment lesson that will be implemented in class. The excerpt begins when the discussion comes to the use of Multiple Choice Test (MCT) for formative purposes. The boundary object is progressively built through the inferences made on its components: on a didactical level, when the actor (teacher or researcher) is conceiving a component of Formative Assessment as a tool for teaching; and on a meta-didactical level, when the actor (teacher or researcher) is conceiving a component of Formative Assessment as an object itself. Teachers discuss about multiple choice test and in particular they would like to propose to students the same test, they have just taken with clickers, in an open written form (as they were used to do before) to see if there is any difference between the two modalities of assessment.

28. T1: […] Does technology do this? Does it lead to different answers? This is the question we had.
29. R1: Uhmm, uhmm
30. T1: Just to have,… why not?
31. R1: Here the difference is between a MCT and an open question
32. T1: Yes… otherwise we should present it...
33. T2: They should have memorized all the options to be in difficulty in the written test
34. R2: No, it is the same question you asked, there is a different process between an open question and a choice among three options that you have to...
35. R1: but it has nothing to do with technology
36. R2: Yes, it has nothing to do with technology
37. T2: No, it is something else
38. T1: It is the modality of assessment.
39. T2: we need to be aware that we don't ask them the same thing. It is true that the tool gives things for the adult, it gives a quick reading, a quick view of the results, by item, per student, but after they …
40. T1: Ah well they, they said to me all the same, it's so good, because suddenly, we do not have to write, it goes faster, …
41. T2: there is still the task of writing that...
42. T1: Oh yes, the task of the writing, ah! this is the first thing they said to me: ah, we could do this every time (laughs) because suddenly, they have a sense of speed compared to assessments where they spent time, yet, they spent time there
43. T2: already, if we keep the pleasure on new concepts like that which in general tired because it's all new, everything …
44. R1: and at the same time, a remark that I wanted to make is not too much related to the assessment, to the summative assessment that you will be able to do, finally, what I would rather like is that we think to look at how we can ensure that students who have not been successful in certain things, how we can get them to succeed, you see …
45. T1: uhm ... but that's our goal today

At a first reading, we can separate this excerpt into different episodes distinguishing the components of the boundary object "Formative assessment" at stake, MCT and technology, as follows:
28-30: the relationship between the two objects (MCT and technology) is unclear, the discussion is imprecise.

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2 FaSMEd: Formative Assessment for Sciences and Maths Education; The research leading to these results has received funding from the European Community's Seventh Framework Programme fp7/2007-2013 under grant agreement No [612337].

31: attempt to distinguish MCT and technology. The proposal is to call in question techniques of formative assessment: MCT and Open question, forgetting at first the way to implement these techniques.
32-34: coming back on the conceptions. The component of the boundary object at stake are still MCT and Open questions. The action is at a syntactic level, allowing to distinguish clearly what is the purpose of the discussion: “*there is a different process between an open question and a choice among three options*”.
35-36: focus on the fundamental difference between use of MCT and use of technology. The action on objects is still at a syntactic level, separating clearly the technique (use of MCT) and the media (technology).
37-43: focus on the modality of assessment. There is an internalization of the distinction between the technique and the media. “37 *No, it’s something else.* 38 *It is the modality of assessment*”. The next remarks show a new type of action on the boundary object: “39 *we need to be aware that we don’t ask them the same thing.”* The distinction in terms of vocabulary is now related to the task itself and its didactical consequences. The action is more on the sense of using one or the other of the techniques with a reflection on the consequences; the action is here related to the meaning.
44-45: focus on global strategies of formative assessment. After the actions of transfer and translation that allow actors to speak of the same object, with a common meaning, the boundary is proposed to be extended in order to be closer to formative assessment process: to take information, to analyze information and to give a feedback to the learners. It is still an action of translation that occurs allowing an extension of the boundary space and a process of internalization.
In our opinion, this excerpt is a good illustration of a beginning of internalization which is led firstly by an action of transfer, an action of translation and finally a beginning of transformation on the component of boundary object regarding the modalities of assessment. The transfer is related to the first distinction made at a syntactic level: the distinction between MCT and technology and the place of MCT within the “container” Formative Assessment. In other words, actors need to clarify the vocabulary to clarify the objects themselves. The final intervention leads to a discussion on the modalities of assessment that is to say on the meaning of the different modalities of assessment in a semantic approach. The two final interventions are the beginning of a pragmatic approach of this internalization that led to the creation of a new shared knowledge about the modalities and functions of formative assessment in the mathematics classroom.
The whole action of transformation is not perceptible in the excerpt itself but in the following of the dialogue and the actual implementation of the formative assessment lesson in the classroom taking into account clearly the distinction between media and the technique, but also placing this reflection in the more general context of the formative assessment, which is precisely the boundary object on which the group works. The properties and functions built on the pragmatic construction of a formative assessment lesson taking into account the MCT in a digital environment allow to infer general properties of formative assessment that contribute to enlarge the understanding of the object both for teachers and researchers: teachers learn a theoretically based definition of Formative assessment while researchers learn the limits of the acceptability in a given institution of the theoretical construct.

Conclusion

The first results that emerge from these analyzes encourage us to continue this work. On the one hand, the framework makes it possible to highlight points that are particularly useful for the analysis and the design of teaching situations. The example of FaSMEd shows that in a particular context, say formative assessment, the collaborative work allows to highlight some properties of the boundary object that would not have been highlighted by one or the other

community. Particularly, the cross-fertilization of the theoretical point of view brought by researchers and the professional point of view defended by teachers brings to light the necessary awareness of strategies of formative assessment and properties of the used technology. Through the boundary objects highlighted in the construction of a sequence, the dialogue enriches the perception of each of the parts and modifies the praxeologies relating to a type of task by internalizing concepts or methods from other communities. On the other hand, this framework also makes it possible to analyze and theoretically justify design-based research in what it proposes as professional development and as theoretical enrichment.
In terms of method, it seems that the analysis of the components of the boundary object at stake and the analysis of interactions on these components highlight the conceptions of the different actors and show the different insights provided by each community. These analyses make it possible to link the macro phenomena of internalization with micro phenomena resulting from local interactions.
In terms of theory, it shows that there is a coherence between the boundary objects theory and the Meta Didactical Transposition and that combining the two approaches highlights the local and global phenomena.
And finally, this theoretical and methodological construction is also useful to design lessons in a perspective of Design Based Research.

References

Artigue, M. (1992). Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. In Dubinsky, E. & Harel, G. (Eds.), *The Concept of Function – Aspects of Epistemology and Pedagogy* (25, pp. 109 – 132). Washington, DC : The Mathematical Association of America.

Arzarello, F., Robutti, O., Sabena, C. Cusi, A., Garuti, R., Malara, N., Martignone, F. (2014). Meta-didactical transposition: a theoretical model for teacher education programmes in Clark-Wilson, A., Robutti, O., Sinclair, N. (Eds) *The mathematics teacher in the digital era*, Springer Science+Business Media Dordrecht.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The journal of the learning sciences*, *2*(2), 141-178.

Carlile, P. (2004). Transferring, Translating, and Transforming: An Integrative Framework for Managing Knowledge Across Boundaries. Organization Science, 15(5), 555-568.

Chevallard, Y. (1982). Pourquoi la transposition didactique ? Communication au Séminaire de didactique et de pédagogie des mathématiques de l'IMAG, Université scientifique et médicale de Grenoble. *Actes de l'année 1981-1982*, 167-194.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherche en Didactique des Mathématiques*, 19(2), 221-266.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5-8.

Lewin, K. (1946). Action research and minority problems. *Journal of social issues*, *2*(4), 34-46.

Star, S.L. (2010). This Is not a boundary object: reflections on the origin of a concept, *Science, Technology & Human values,* Vol. 35, n° 5, pp. 601-617

Star, S.L. & Grisemer, J.R. (1989). Institutional Ecology, `Translations' and Boundary Objects: Amateurs and Professionals in Berkeley's Museum of Vertebrate Zoology, 1907-39. *Social Studies of Science*, 19(3), 387-420.

Swan, M. (2007). The impact of task-based professional development on teachers' practices

Designing Interdisciplinary Learning Units Drawing on a Research Project Aimed in Assessing Mathematics Teachers' key Competencies

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Abstract: The aim of this paper is to introduce and discuss some indicators to evaluate mathematics teachers' competences drawing on the discussion of the design of an interdisciplinary *learning unit* (LU) embedded in the Professional Training for Pre-K and Primary pre-service teachers program, in a HE (High Education) program for mathematics teachers, drawing on a *design research* approach. We conclude that this experience provided insight for pre-service teachers to develop some fundamental competencies concerning teacher training abilities.

Résumé: Le but de cet article est d'introduire et de discuter de certains indicateurs permettant d'évaluer les compétences des enseignants de mathématiques, en s'inspirant de la discussion sur la conception d'une unité d'apprentissage interdisciplinaire (UL) intégrée dans le programme de formation professionnelle des enseignants avant l'enseignement primaire et primaire, dans un programme d'enseignement supérieur pour professeurs de mathématiques, en s'appuyant sur une approche de recherche en conception. Nous concluons que cette expérience a permis aux enseignants en formation de développer certaines compétences fondamentales concernant les capacités de formation des enseignants.

The aim of this paper is to introduce and discuss some indicators to evaluate mathematics teachers' competences drawing on the discussion of the design of an interdisciplinary *learning unit* (LU) embedded in the Professional Training for Pre-K and Primary pre-service teachers program, in a HE (High Education) program for mathematics teachers (Castelló, Giménez, Godall, Puig, & Tilló, 2017). This work is framed within an ARMIF[3] research project. We were also drawn on some contributions from the RTD research project (EDU2015-64646-P) on the development of the competence of didactical analysis. Introducing multi, inter and transdisciplinarity (Bourguignon, 1997, D'Ambrosio, 2011, Piaget, 1972) within HE programs is a current trend due to the international discussion on STEM. Recent studies have been discussed the need to facilitate and support the professional development of the teaching staff using new curricula designs (Gast, Schildkamp, van der Veen, 2015). Although most of the previous work in the field has focused on traditional types of professional development interventions, the current situation due to the STEM debate (Johnson, Peters-Burton, & Moore, 2015) forces us to reconsider the approach of the professional development for future teachers that we are offering in our HE organizations. In his paper, we first introduce the main aim of our research study. Then, we highlight the research questions that we want to discuss with the audience here. We provide some methodological details. Then, we discuss our data and provide a set of indicators, as a draft to be discussed, to identify and to assess the development of key competencies to evaluate teachers of mathematics' key competencies.

[3] ARMIF is a grant funded by the Catalan Government aimed at improving professional teacher training programs. See further information at https://bit.ly/2uisz4f

Aim and main idea

There is an increasing interest in looking for avenues of dialogue between researchers and teachers to improve teachers' teaching effectiveness. During the last decades, experiences such as *lesson study* (Fernández & Yoshida, 2012) or teachers acting as researchers (Muijs & Reynolds, 2017) have been the more and more common all over the World. Even successful organizations such OECD have included the teachers' perspective within their surveys, as demonstrated by the delivery of TALIS (Teaching and Learning International Study) since 2008. Professional development appears to be a relevant aspect of teachers' effectiveness in teaching mathematics (Díez-Palomar, 2017). However, as stated by Kennedy (2016), not all professional development initiatives promote effective learning. To avoid the adverse effects of adopting ineffective actions, we need to base the design of our professional development programs on solid and reliable research findings.

In this paper we address three main research questions:

1) In what way does the dialogue between disciplines make possible the enlightening of coherence of learning for students? How does this dialogue strengthen the teaching and learning of mathematics?
2) How does interdisciplinarity contribute to the creation of concepts specific to the mathematics discipline?
3) What indicators can we identify to assess the development of key competencies in a program of teacher' professional development?

Methodology

A *design research* approach was adopted in this study (Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). Five different courses were involved: Mathematics, science and education, Knowledge and exploration of the natural environment, Music and body expression, Educational systems and school organization and Science teaching and learning. In this paper, we drew on the work done within the course "Mathematics, science, and education." Participants were pre-service teachers. Five in-classroom non-sequential sessions were conducted along two months, as well as group work out-of-classroom (see table 1). Students worked in small groups. They selected an article in a newspaper because being interesting regarding science and mathematics. Drawing on this focus, they had to create a lesson unit integrating both mathematics and sciences, addressed to pre-K and K students.

Table 1. Sessions' designed structure.

Session 1: Introduction to the ARMIF framework. General aspects and specific instructions. Session 2: Introducing the selection of articles within newspapers. Discussion of the tentative concepts that may be used to plan the lesson unit. Session 3: Design of a classroom session. Draft. The professor discusses the draft with each group of students. Sessions 4 and 5: Presentation of the final version of the classroom session designed and the results if implemented. Self-evaluation and directed discussion with the whole group. * After the last session, each group of students presents their final assignment, adding the contributions from the presentation in session 5.

In each lesson unit some key competencies were assessed at least in two different moments (at the beginning -session 1- and during the design -session 2 or 3-). All sessions included *time for reflection* were participants from each small group had to identify their progress in designing their lessons. At the end (sessions 4 and 5) all participants analyzed their work regarding the planned lesson qualitatively. Students' presentations and focus group discussions were used to conduct such an analysis. Each group of students presents their work to the mainstream. In such a presentation, the future teachers present not only the tasks for early childhood children but also the Science and Math approaches and their relationship. Some different chosen topics are challenging and original for Kindergarten's children.

Figure 1. Images are given by the group presenting the theme "Archeological time and human evolution."

After hearing each of the projects, all students (as well as the teacher) used a rubric to evaluate their peers' (students) work. They discussed what kind of indicators can we identify to assess the development of teachers' key competencies.

Discussion and conclusions

All students were able to design a lesson integrating mathematics and science in their designs. They combined both components (mathematics and science) to produce innovative STEM lessons (e.g., the Solar System, the Volcanoes, the Earthquakes, the Honeybees, etc.) addressed from Pre-K to Primary pupils. Using this "dialogue" between Mathematics and Science the pre-service students were able to create situations and contexts for effective mathematics learning.

One aspect that pre-service teachers think that is crucial to plan their didactic sequences is the contextualization. Thus, for example, we find statements such as:

> *"In particular, we have started with a piece of news to be able to explain a set of mathematical contents, so that all students would be motivated, and the tasks are contextualized in a familiar and real context." (Group "The marine turtles").*

On the other hand, we find reflections that, in addition to highlighting the importance of contextualization, allude to the need to address innovative and complex issues in early childhood education:

> *"On the one hand, the choice of the text from a newspaper allows us to deal with a breaking news topic, which usually is not included as a lesson in the regular classroom in early childhood education; but we believe that it is possible to use it satisfactorily if we adapt it to that age stage. Besides, that piece brings us closer to the discovery of a new species drawing on the bones found in South Africa. This fact has allowed us to devise activities that relate the adaptation and evolution of species with their physical and functional changes, specifically the bones and the use of different instruments to build tools." (Group "Archeological time and human evolution").*

In the example of the study about evolution, we analyzed that the group proposes to discover a treasure of archeological materials (different bones, stones, glasses to travel through time, and so on). They planned a task addressed to 5 years old children, asking them to classify the shapes of the archeological pieces, putting them on an evolution table representing the evolution timeline. The following tables 2 and 3 summarize the indicators developed by the research team to evaluate students' contributions. Table 2 is focused on the evaluation of the competence "planning and management of the lesson."

Table 2. Indicators to evaluate the competence "Planning and management of the lesson."

Dimensions	Categories	Indicators
Structure and strategic thinking	The relevance of the structure	Analysis-synthesis: Coherence, correct identification of the key ideas
	Strategic action	Integration of structures and systems of meaning. Use of schemes to explain new relations of knowledge
	Complexity	Addressing complex situations
Adaptability and regulative control	Re-addressing	Be able to design proposals to improve
	Self-criticism	Learn and improve from the own mistakes
	Transference	Face new situations
Decision making and applicability	Coherence	Exchanging ideas in complex situations, in specific moments of the work timeline
	Capacity of reaction	Acceptance of the mandatory changes to re-address the situation
	Applicability in real situations	Carrying out the proposed methods

In the evaluation that the students carry out on their work and one of their peers, they recognize different aspects as presented in Table 2. Statements such: "*The report has addressed a whole series of aspects related to science and logic-mathematics fields, such as classification, comparison, relationships, measurements, logical reasoning, time and change, human beings, the discovery of natural elements in the environments, among others.*" (Group "Archeological time and human evolution"), together with the own design of the activities addressed to the children, are evidence of the pre-service teachers' ability to recognise the category: *relevance of the structure*. This happens because pre-service teachers can identify key ideas and coherence among the proposals.

Table 3, in turn, is devoted to analyzing the competence "Creativity, management and implementation."

Table 3. Indicators to evaluate the competence "Creativity, management and implementation."

Dimensions	Categories	Indicators
Creative learning	Innovative opening	Generation of new situations of ideas. Proposal of breaking ideas regarding the established procedures

	Flexibility	Using different kinds of strategies
	Originality	The contribution of original ideas to the problems presented based on the resources that are known
	Integration and formal expression	Appropriate approach to generate ideas
	Creative application	Recognition of innovative ideas
Management capacity	Project design	Prioritization of mid-term and long-term objectives, undertaking corrective actions if necessary. Flexible planning and coordination of collaborative tasks
	Global management	Application of quality levels. Analysis of the context to define specific objectives as a response to innovative challenges that s/he proposes.
Entrepreneurial and innovative capacity	Leadership	A global vision of the reality that surrounds him an evaluation of positive and negative aspects of the innovative proposals
	Implementation	Active positioning and ability to convince. A proposition of improvements needs
	Analysis of risks and benefits	Planning possible actions that will be undertaken

An interesting example is a work about "Swirls." Since the beginning, the future teachers engaged in the activity asked the trainer: "*We think it is an interesting theme for Kindergarten but, we do not know how to introduce mathematics in this theme?"* After several discussions, they finally discussed the spiral shapes in three dimensions, which is unusual for activities addressed to children of this age. Such an approach means the reflective process to understand the need for adapting mathematical knowledge to be understandable without forcing the mathematical objects.
In addition to the recognition of the possibility of tackling unusual topics, the group of pre-service teachers that proposed the theme "swirls" implemented one of the activities of its didactic sequence with some children. Next, we introduce a part of the episode. In the discussion after the implementation of this activity carried out with the trainer, the group of pre-service teachers recognizes the importance of dialogue to produce the hypothesis.

Activity: How does the water go down the sink?
The teacher has placed the cap on the sink, and she filled it with water, posing the following questions:

FM What will happen if I remove the cap?
A1: Water will run.
FM: Good! It will run. So, where will it run?
A2: It will go off the cap.
A3: It will go through the hole.
What you mean it would go through the cap?
A3: If the cap comes out the water will go there, through the holes.

FM: Then the cap is covering the holes in the sink so that the water does not run away, right?
A1: Yes.
FM: Then if we remove the cap the water will run away. How do you think it's going?
A4: Like a slide!
A2: Like when you wash your teeth.
FM: So does the water make some way when it goes?
A1: It's like laps.
FM: Like laps? OK, make a turn like those that the water makes with the finger.
A1: Like this? (she draws a circle with the finger in the water)
FM : Does anyone want to draw another shape with the finger?
A6: No, because you cannot do with the finger what the water does…
A7: Yes! The water goes like this (draws a spiral).
FM: I see that you know a lot, let's see what happens… You must pay attention (The teacher removes the clap, and she restores it when half of the water is left). What happened?
A1: It did laps, many laps!
FM: Can you draw the laps in the water? (A1 draws a spiral) Great job! It's getting smaller and smaller laps, you see it?
FM: To which side did the water rotate?
A7: Like this (points out to a direction with the finger)
FM: And now, if I remove the cap again, where will it rotate? Towards the same side or the other?
A8: Now it's the other.
A5: It depends on whether you remove the cap with one hand or the other.
FM: Let's try! Look closely (She removes the cap with the right hand and covers it; wait for a few seconds and repeat it with the other hand).
A2: It turned the same! Every time is the same.

The group of pre-service teachers identifies that children recognize a flat image of the phenomenon (the spiral), although with their finger they insinuate a 3-D figure. They realize that children correct their hypothesis.
We also found reflections suggesting the progress of the pre-service teachers in understanding the dimension "Entrepreneurial and innovative capacity," which demonstrate their capacity to differentiate proposals and evaluate positive and negative aspects of the innovative proposals.

Regarding the work of our peers, we have to say that their topics are very different ones from the other. From our point of view, some of them took the risk dealing with very complex topics, in the sense that they need to work harder to teach them, so the students would be able to understand them. Other ones did not choose topics as complex as those ones; however, the activities that they designed are very original, creative and appropriated." (Group "Healthy eating and active life").

Finally, we want to conclude with the words from one of the students participating in this study, regarding his perception of the experience:

I think that the proposed experience is very useful for a P-K professional training lesson, since it provides them with elements that enhance each of the capacities that must be developed at that stage (this is an activity that promotes the child's integral development):

autonomy (when carrying out an information search and developing your own map design); communicate and inhabit the world (through cooperative work - it implies constant negotiation and coexistence with colleagues and forces them to maintain attitudes of respect towards the opinions of others - that this methodology requires thanks to the contributions they receive from their peers, which can provide different points of view and knowledge that can cause them to rethink their own way of understanding and deal with the issue)

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References

Bourguignon, A. (1997). De la pluridisciplinarité à la transdisciplinarité. *Congrès de Locarno, 30 avril-2 mai 1997. Annexe au document de synthèse UNESCO* http:://ciret-transdisciplinarity.org/locarno/loca5c1.php

Castelló, J., Giménez, J., Godall, T., Puig, J., & Tilló, T. (2017). Unidad de aprendizaje 2. En V. Violant Holz (Ed.), *Unidades de Aprendizaje Interdisciplinar en la Doble Titulación de Maestro de Educación Infantil y de Educación Primaria* (pp. 53-77). Barcelona: Octaedro – ICE UB.

D'Ambrosio, U. (2011). *Universidades y transdisciplinariedad*. Visión docente Con-Ciencia, X(58) 5-17.

Díez-Palomar, J. (2017). Nuevas aportaciones para la mejora de la enseñanza de las matemáticas. *REDIMAT*, *6*(1), 3-5.

Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Routledge.

Gast, I., Schildkamp, K., & van der Veen, J. T. (2015, August). Professional development in the context of a higher education curriculum innovation. In *EARLI 16th biennial conference: Towards a reflective society: synergies between learning, teaching, and research*.

Johnson, C. C., Peters-Burton, E. E., & Moore, T. J. (Eds.). (2015). *STEM roadmap: A framework for integrated STEM education*. Routledge.

Kennedy, M. M. (2016). How does professional development improve teaching?. *Review of Educational Research*, *86*(4), 945-980.

Muijs, D., & Reynolds, D. (2017). *Effective teaching: Evidence and practice*. Sage.

Piaget, J. (1972). L'épistémologie des relations interdisciplinaires. *In OCDE, L'Interdisciplinarité, problème d'enseignement et de recherche* (pp. 131-144).

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.). (2006). *Educational design research*. Routledge.

"The mathematician is present". Report from a scientific Mini-Residency at Bode-Museum Berlin

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Abstract: This paper is about detecting examples of mathematisations at an art museum and is part of a workshop designed to provide a meeting space between art and science. This workshop is the result of a science-residency at the Bode-Museum and is based on theoretical ideas stemming *from critical mathematics education*. It starts with Ole Skovsmose's distinction between different subject roles in the context of "mathematical modelling": constructors, operators and consumers. In the classroom, the focus is on operating and consuming mathematics. This project, however, takes the out-of-school experience for teachers and students as a starting point from which to look at the role of the constructors in the museum. In effect, this amounts to engaging in a form of "mathematical archeology". As the project is only in its initial stages, I would like to present some of the ideas at this conference, in order to engage in a discussion and share the experiences of addressing the process of categorisation and classification as a mathematical activity in this "interdisciplinary dialogue".

Résumé: Cet article traite de la détection d'exemples de mathématisations dans un musée d'art et fait partie d'un atelier conçu pour fournir un espace de rencontre entre art et science. Cet atelier est le résultat d'une résidence en sciences au Bode-Museum et est basé sur des idées théoriques issues de l'enseignement critique des mathématiques. Cela commence par la distinction faite par Ole Skovsmose entre différents rôles de sujet dans le contexte de la «modélisation mathématique»: constructeurs, opérateurs et consommateurs. En classe, l'accent est mis sur l'utilisation et l'utilisation des mathématiques. Ce projet s'inspire toutefois de l'expérience extrascolaire des enseignants et des élèves pour examiner le rôle des constructeurs dans le musée. En réalité, cela revient à s'engager dans une forme «d'archéologie mathématique». Le projet n'en étant qu'à ses débuts, j'aimerais présenter certaines des idées de cette conférence, afin d'engager une discussion et de partager les expériences de la catégorisation et de la classification en tant qu'activité mathématique dans ce «débat interdisciplinaire». dialogue".

Introduction

Art education, especially museum education, is a relatively young field in Germany, and there are few opportunities to gain vocational qualifications. In order to strengthen and expand the vocational side of this field, an initiative was launched by the Federal Cultural Foundation and the National Museums in Berlin. The name given to this initiative, *lab.bode,* describes a laboratory-style space, affiliated to the Bode-Museum on the museum island, in which a team of people experiment with different formats of education in the arts. This will run till 2020.
It was in this context that I was invited as a maths education researcher to search for links between mathematics and the museum; links which are specifically related to this collection but which could also be applicable to other museums in the future. The idea is to provide an impulse that will enrich the work there, in terms of mathematics and which can then be used to develop further workshops. The exceptional learning environment of the museum complements the teaching of

mathematics in German classrooms, especially the mathematics dealing with the social and cultural phenomenas and processes. Inspired by the current exhibition at the moment entitled *„Beyond Compare: Art from Africa in the Bode-Museum"*, the topic of comparing and establishing categories soon came to mind as a connection between mathematics and museum work and one which would be interesting to explore more deeply: comparing and categorizing.
I would now like to illuminate the theoretical background stemming from the link between mathematics education and democracy (Skovsmose 1998) and present some examples and initial sketches for potential workshops.

Mathematical modelling in the art work
'I don't know anything about mathematics. Mathematics and me? We are enemies.' This was the kind of response I heard frequently from many people during my residency: from scholars in the theory and history of art, artists, art educators and curators. They seem to reflect the assumption of Eva Jablonka, that most students or people meet applied mathematics in its existing form, as 'implicit mathematics' (Jablonka 2010, p. 94f.). We accept, for example, the standardisation of time, space, money or the division of labour time as abstract matters of everyday life, without recognising them as mathe-matisation (also known as *realised abstractions,* see Keitel, Kotzman, Skovsmose 1993, p. 255f.). The theoretical approach of this project has been chosen on the assumption that daily classroom activities provide little space and time for project-based discoveries and reflections on the social and personal impact of mathematics. Based on an analysis of teaching materials and proposals presented at teacher education conferences, Jablonka (e.g., 2010) noted the lack of opportunities for engaging in reflections on the perspective of mathematical models constructers and thereby being able to grasp the fundamental decisions involved. How could we then explore this in an area in which we do not expect to meet mathematics: an art museum?

The distinction made by Ole Skovsmose (Skovsmose 2006) between the social groups which are involved in or affected by mathematics could be helpful. He distinguishes between *constructors, operators and consumers* The *constructors* are those, who 'develop and maintain the apparatus of reason' (p. 140) and have power over the operators and consumers by constructing social and material technology while the *operators* are in the position of making decisions only on the input and output of these technologies. Operators in a museum could be those employees who feed databases with information exhibits for a particular collection or exhibition in the museum. The *consumers* are only confronted with the output of these complex models, which they meet in forms of reports and statements, often as figures and statistics. While justifications of decisions in newspapers or reports are based on this data, the consumers are unable to access the processes which lead to generation of this data. If there is no communication between the consumers, operators and constructors, or feedback given to the authorities affiliated with the constructors, then we have an implicit social hierarchy. Skovsmose talks about a democratic attitude, where inaccessibility and invisibility of mathematical models are 'a threat to participatory democracy' (Skovsmose 1998, p. 198). Since a museum is perceived as an exclusive space, mainly addressed to people from the 'educated middle class', this perspective on mathematics contributes to the goals of the *lab.bode,* which are to open the museum to people from different parts of society. The visitors to an exhibition can be seen as consumers of the mathematisations reflected in the exhibition, whereas the curators are the constructors. Although categorization is an

pic. 2 *The Virgin and Child*
Donatello

activity that goes beyond mathematics, it is both typical and fundamental to mathematical action. As Skovsmose claims, that mathematics can also be interpreted as 'a source of decision-making and action' (Skovsmose 1998, p. 196) and these decisions are also justified on the basis of mathematical arguments. I therefore use the museum as an opportunity to bring to light the constructor's role with the help of *mathematical archeology* (Skovsmose 1998).

Examples of different categorisations in art exhibition

I would like to mention two examples in which the exhibits are arranged in a way that is different to usual (according to geographical or political areas and epochs):

pic.1

One is a room in which artworks are arranged around the Pazzi Madonna by Donatello (1386-1466). The Pazzi Madonna is one of the masterpieces by the Italian sculptor and an emblem of early Renaissance culture copied by many artists (see Rowley 2016). It illustrates the aspiration of artists from that time to create a convincing and real life experience. The Virgin and Child are shown in life-size and framed in a niche which follows the rules of linear perspective: the receding lines of the niche converge at one single point. As a result, both figures seem to enter into the world of the viewer, an impression which I also gained while kneeling in front of the Madonna. This example of mathematical constructions in art is used to convey a certain experience to the viewer. In this small room, in which the Pazzi Madonna hangs in the center, numerous other Madonnas are arranged around it, some of which are also by Donatello himself or others by artists who have copied his style. The larger room preceding the Donatello room contains other works of art, sculptures and paintings of Mother Mary with Child (pic.1)

pic.3 ornamental stripe of a tunic,
Egypt, Faijun, 6/7th cent.

pic. 4 exhibits in room 255
South of Germanx, Baroque

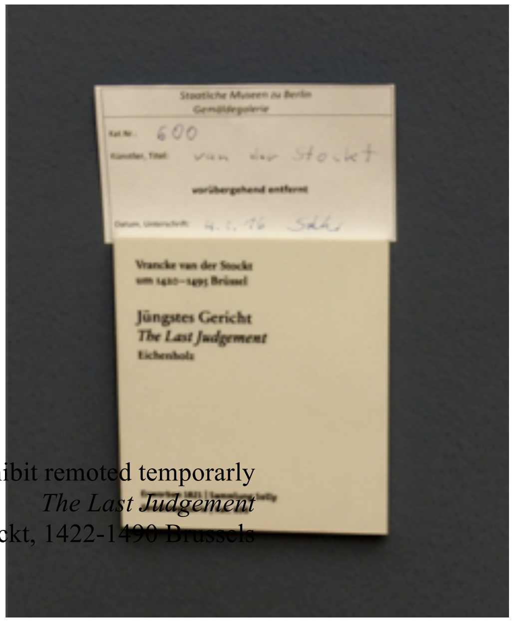
Another example is the rare textile collection at the Bode-Museum. Whereas, in the former example the objects are arranged in space according to their content, the Byzantine art room uses a system based on the material. As textile is an organic material that requires special conditions such as low light, the textiles are stored in individual drawers one above the other. And the exhibited textile pieces are only fragments of a garment, which is reconstructed by archaeological and art historical research (pic.3) These two examples of different categorisations of pieces of art and objects in an exhibition are the result of an extended process. A process of comparison and decision making with different impact on the insights that may follow for the visitor. To detect this might contribute to the insight the curator Maria Lopez-Fanjul came to in an interview: "If mathematics is not only numbers and calculating, but also logic, than I am a very mathematical person."

<u>Potential Workshops</u>Viewing the exhibitions only gives us the tip of the iceberg, so to speak, in discovering the extend of mathematics in the museum, in the form of basic categorisations and classifications used for counting and quantifying the exhibit. Since the museum is also a state institution, *mathematical archeology* touches on many processes involved when 'excavating mathematics which might be encapsulated in certain political arguments, technologies or administrative routines' (Skovsmose 1998, p. 199). The aim of this workshop would be to gain an idea of the conditions and possibilities involved in developing exhibitions or working in an art museum and to realize them mathe-matically. First we plan to start with looking at the exhibits and finding the common attributes according to they are arranged (pic. 4).

The students also have the possibility to arrange them according to other categories.

A missing exhibit gives the starting point for rising questions, where this object went to and why etc. (pic. 5), though the students come in contact with the institution museum.

At first there will be a series of workshops in November 2018 to develop the concept of the workshop for 2019. These series are constantly evaluated to improve to methods and content concerning the connection between the logic in mathematics and the logic behind exhibitions. The research questions faced with a kind of action research are: „How can we find common ground between categorization as a proto-mathematical activity and the categorization of art exhibits? Which challenges in communication to the students and teachers might arise in the workshops? What are the approaches to overcome them?"



pic.5 missing exhibit remoted temporarly
The Last Judgement
Vrancke van der Strockt, 1422-1490 Brussels

What do we understand as mathematics?

Numerous conversations during the residency showed the need to define the understanding of mathematics in this project. Above showed quotations reveal, that mathematics is very often seen as "a scary ghost from old times", so to say as an object, which exists somewhere, mostly in the mind of humans, "representing an objective truth". Whether this is so, I can not discuss here, but I would like to refer to the arguments of constructivism as well as the fact that even mathematical concepts are a result of construction through comparison. My theory is that it is important to become aware of the point of comparison that configures the category in order to decide which concepts and operations it makes sense to use (Hersh/Davis 1985, p. 69f.). In showing the instability and development of mathematics, in contrast to the widely unquestioned constancy of mathematical systems, the aim of this project is to provide a space in which mathematics can be experienced in another way. This means that we can either look for parts of the museums where there are already existing examples of mathematisations used, such as numbers and measurements or new fields can be discovered for examples through comparison and category building.

This creates room and reason for mathematical thinking and the disembodiment of mathematics is given a living body, who experiences the (de-)construction of knowledge (Straehler-Pohl et. al. 2018, '9''05; see also Kremling et al. 2018). The aim of the workshop is to establish that 'somehow everything could be connected to mathematics'. The insight about mathematisation, gained through the experience of the workshops should be reflected in discourse and links to (classroom) mathematics by the math teachers. As a result, students are not left with an understanding of mathematics as a distance and fixed subject, but can relate to their own experiences and see it also as a social process.

Literature

Jablonka, Eva (2010): „Reflections on mathematical modelling" In: Helle Alrø; Ole Skovsmose (Hrsg.): *Critical mathematics education : past, present, and future. Festschrift für Ole Skovsmose.* S. 89-99. Rotterdam.

Davis, Philip J./ Hersh, Reuben (1985): Erfahrung Mathematik. Springer. Basel.

Jablonka, Eva (2010): „Reflections on mathematical modelling" In: Helle Alrø; Ole Skovsmose (Edit.): *Critical mathematics education: past, present, and future. Festschrift für Ole Skovsmose.* S. 89-99. Rotterdam.

Keitel, Christine; Kotzmann, Ernst; Skovsmose, Ole (1993): „Beyond the Tunnel Vision: Analysing the Relationship Between Mathematics, Society and Technology", In: Keitel, Christine; Ruthven, Kenneth (Edit.): *Learning from Computers: Mathematics Education and Technology.* Basel/Berlin.

Kremling, J., Arndt, N., Roman, A., Zilm, M., & Straehler-Pohl, H. (2018). Dancing on the junction of mathematics and bodily experience. *The Mathematics Enthusiast, 15*(1), 201-227.

Rowley, Neville (2016): *Madonna und Kind (die Pazzi Madonna) / The Virgin and Child (the Pazzi Madonna).* Description in SMB-digital. Online collections data base. http://www.smb-digital.de/eMuseumPlus?service=direct/1/ResultLightboxView/result.t1.collection_lightbox.$TspTitleImageLink.link&sp=10&sp=Scollection&sp=SfieldValue&sp=0&sp=0&sp=3&sp=Slightbox_3x4&sp=0&sp=Sdetail&sp=0&sp=F&sp=T&sp=10 (24.05.18)

Skovsmose, Ole (1998): „Linking Mathematics Education and Democracy: Citizenship, Mathematical Archaeology, Mathemacy and Deliberative Interaction". In: Zentralblatt für Didaktik der Mathematik (1998) 30: pp. 195-203. https://doi.org/10.1007/s11858-998-0010-6 (24.05.18)

Skovsmose, Ole (2006): *Traveling through education: Undertainty. mathematics. Responsiblety.* Rotterdam.

Straehler-Pohl, H., Roman, A., & Kremling, J. (2018). „Asking mathematics for a dance - Utopian mathematics in bodily performance?" *Performance at the DGfE-Kongress 2018.* https://vimeo.com/261556796.

À travers des lentilles islamiques: matériaux pour une histoire des aides visuelles au service de l'enseignement des mathématiques

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Abstract: Arab texts still very little known, and which have not yet been exploited for a universal history of visual aids, testify to the knowledge of magnifying lenses for reading, in the Islamic civilization, from the tenth century. The prehistory of optical aids, especially glasses, which appeared around the end of the 13th century, would thus be buried in Arabic sources until now. Another unexpected result: the study of optics resulted in the construction, by an Ottoman scholar of the late sixteenth century, of an instrument functioning as a telescope. This work, which is part of the history of science, could be used in the context of an interdisciplinary teaching linking at the same time Mathematics, Physics and History of Sciences where it would be an important factor of motivation of the pupils in difficulty or lacking motivation in mathematics class, giving meaning to their teaching. The history of science contains a multitude of pedagogical applications and the history of visual aids, being related to everyday life, would be a convincing example. This work, which constitutes a new deal in the history of science, namely the opening of a new field of knowledge and investigation, also has an innovative potential through its application to the didactics of mathematics.

Résumé: Des textes arabes encore très peu connus, et qui n'ont pas encore été exploités pour une histoire universelle des aides visuelles, témoignent de la connaissance des lentilles grossissantes pour la lecture, dans la civilisation islamique, dès le Xe siècle. La préhistoire des aides optiques, notamment celle des lunettes, apparues vers la fin du XIIIe siècle, se trouverait donc jusqu'à présent enfouie dans des sources arabes. Autre résultat inattendu: l'étude de l'optique a abouti à la construction, par un savant ottoman de la fin du XVIe siècle, d'un instrument fonctionnant comme un télescope. Ce travail, qui s'inscrit dans l'histoire des sciences, pourrait être utilisé dans le cadre d'un enseignement interdisciplinaire reliant à la fois Mathématique, Physique et Histoire des Sciences où il serait un facteur important de motivation des élèves en difficulté ou démotivés en classe de mathématiques, en donnant du sens à leur enseignement. L'histoire des sciences recèle une multitude d'applications pédagogiques et l'histoire des aides visuelles, en rapport avec le quotidien, en serait un exemple probant. Ce travail, qui constitue une nouvelle donne dans l'histoire des sciences, à savoir l'ouverture d'un champ nouveau de connaissance et d'investigation, a aussi un potentiel innovatif par son application à la didactique des mathématiques.

Histoire des sciences et mathématiques: une gageure

Cette communication est extraite d'un travail de recherche en cours, à savoir la première tentative jamais entreprise de faire une introduction à l'histoire des aides visuelles dans les cultures d'islam. [1]

Hormis son intérêt évident pour l'histoire des sciences, ce travail pourrait être utilisé dans le cadre d'un enseignement interdisciplinaire reliant à la fois Mathématique, Physique et Histoire des Sciences. Ce dernier volet constitue le noyau sur lequel vont se greffer les deux autres disciplines.

L'objectif est donc double: d'une part présenter des textes très peu connus mais pertinents pour l'histoire des sciences, et, d'autre part, de montrer qu'ils se prêtent à des applications didactiques, offrant des pistes intéressantes à explorer dans le cadre de l'enseignement des mathématiques.

Outre qu'elle constitue un apport culturel, l'histoire des sciences présente des avantages pédagogiques indéniables: elle contribuerait à éveiller, voire aiguiser chez les élèves leur sens de l'histoire; elle est une porte d'entrée et un facteur important de motivation pour l'enseignement des mathématiques et qui lui (re)donnera du sens; en favorisant le «comment» et le «pourquoi» elle permettrait de devancer la question classique des élèves en difficultés ou démotivés durant les cours de mathématiques «à quoi cela sert-il?»

L'histoire des sciences recèle une multitude d'applications pédagogiques [2]; l'histoire des aides visuelles, comme chapitre inédit de l'optique, et en rapport direct avec le quotidien, en serait un exemple probant. Cet article présente trois textes remarquables avec de brèves suggestions d'application pédagogique à explorer et à tester sur le terrain par une pédagogie de groupe (p. ex. en France Troisième-Seconde) ou de projet (Première S).

La préhistoire des aides visuelles: un champs nouveau d'investigation

Dans sa célèbre biographie *Aʿyān al-ʿAṣr fī Aʿwān al-Naṣr (*«Les notables de l'époque et les aides de la victoire), Khalīl b. Aybak al-Ṣafadī (1297-1363) nous a conservé deux vers d'un de ses contemporains (mort en 1355), une sorte d'instantané poétique empreint d'humour et de mélancolie à l'approche de la vieillesse:

Quel chagrin à cause du temps de l'enfantillage
Mais j'étais destiné à ce que mon chagrin grandisse
Mes yeux auparavant se trouvaient sur mes joues
Aujourd'hui ils se trouvent sur mon nez.

Ceci est le premier témoignage connu de l'utilisation de lunettes dans le monde musulman dans la moitié du XIVe siècle. [3]

Quelle est l'origine de cet instrument aujourd'hui si indispensable? Probablement Pise, vers 1286, date présumée de l'invention des lunettes. Mais la chose intrigante est que jusqu'à aujourd'hui personne ne connaît les circonstances de l'invention, ni a fortiori l'avant-1286! Cet anonymat donna naissance à un narratif, ou plutôt à un fatras inextricable de spéculations, de falsifications, de susceptibilités chauvinistes, qu'Edward Rosen, historien de l'optique, finit par démonter pour conclure que cette découverte était due à un profane.[4]

Une première exploration de l'histoire de l'optique fit émerger le célèbre savant al-Hasan ibn al-Hasan Ibn al-Haytham, dit Alhazen, (né en 965 à Basra, mort vers 1038 au Caire) comme étant à l'origine de l'idée des aides optiques.[5] Pour résumer une longue histoire, ni Ibn al-Haytham, ni son successeur persan, Kamaladdin al-Farisi (1267-1319), ne parlent de l'utilité des lentilles comme aides visuelles, alors qu'ils connaissaient fort bien.

Quant à l'ophtalmologie arabe, il ressort d'une étude de 60 médecins renommés entre 800 et 1300 que les aides visuelles y sont inconnues.[6] Tout au plus trouve-t-on dans les ouvrages de minéralogie des conceptions ésotériques, comme la recommandation d'Ahmad ibn Yûsuf at-Tîfâshî (né en 1184 à Tifesh, dans le Nord-Est de l'Algérie, mort en 1253 au Caire), auteur d'un fameux lapidaire, *Azhâr al-afkâr fi jawâhir al-ahjâr* «Les Fleurs des idées sur les pierres précieuses», de fixer une émeraude, pour reposer les yeux fatigués,[7] ainsi que l'avait déjà laissé entendre Pline l'Ancien (23-79) dans la célèbre phrase: «l'Empereur Néron regardait avec une émeraude les combats de gladiateurs».[8] Tel est le bilan désappointant, mais...

Un rubis au Sri Lanka

La Serendipité, qui est la muse du chercheur et sa bonne étoile, car elle lui fait découvrir des choses précieuses qu'il ne cherchait pas, tire son nom de Serendip, l'actuel Sri Lanka, «l'île des Rubis» des géographes musulmans (*Jazîrat al-Yâqût*), et le joyau que je vais évoquer brièvement gisait dans le *Kitâb al-Jamâhir fî ma'rifat al-jawâhir* «Livre sur l'abondance des informations et la connaissance des joyaux», du célèbre et génial érudit centre-asiatique, Abû r-Raihân Muhammad b. Ahmad al-Bîrûnî (né en 973, à Kath, Ouzbékistan, mort en 1048, à Ghazni, Afghanistan). Voici ce qu'il relate:

as-Sallâmî rapporte sur l'autorité d'al-Lahhâm que Abû Bishr al-Sîrâfî étant une nuit chez son oncle maternel à Sarandîb, celui-ci sortit un rubis rouge poli qu'il posa sur les lettres d'un livre pour le lire. Le narrateur en fut étonné considérant qu'il faisait déjà nuit et qu'un objet transparent rayonnait sans qu'une lumière émanant d'un corps lumineux tombât sur lui. Ce rubis était en forme d'hémisphère dont la surface plane était contigüe au livre. Les lignes fines peuvent être lues avec un cristal de roche [*billawr*] de même forme, car l'écriture derrière lui est grossie pour les yeux, et les lignes [l'espace interlinéaire] sont élargies. Les causes en incombent à la science de l'optique.[9]

Ce texte rare, qui remonte au XIe s., mais dont les informants vivaient au milieu du Xe, est la description précise d'un rubis en forme de lentille plan-convexe et de son utilisation comme pierre de lecture. Ceci est pour l'heure le premier récit de l'utilisation des propriétés optiques du rubis et du cristal de roche pour la lecture. Toutefois, il est frappant que ces propriétés optiques si remarquables aient échappé à tant de lecteurs, éditeurs et traducteurs de cet ouvrage. Pour les élèves, ce serait l'occasion d'explorer l'aspect matériel de la production de lentilles (tour, cristal de roche, substances abrasives), puis optique (théorie de la vision, physicalité de la lumière, lentilles ardentes et grossissantes, la réflexion, la réfraction) et mathématique (géométrie). **[Fig. 1 et 2]**

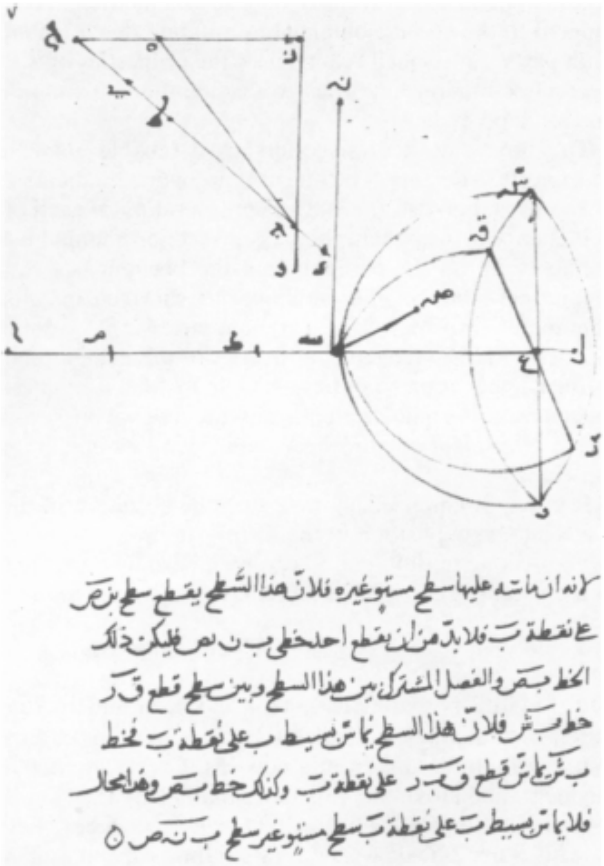

Fig. 1: lentille biconvexe extr. du Traité sur les Instruments Ardents
d'Ibn Sahl, Xe siècle . MS 867, folio 7r (Milli Library, Tehran)
(Voir: Roshdi Rashed, A Pioneer in Anaclastics. Ibn Sahl on Burning Mirrors and

Lenses, ISIS 1990, 81, p. 467)

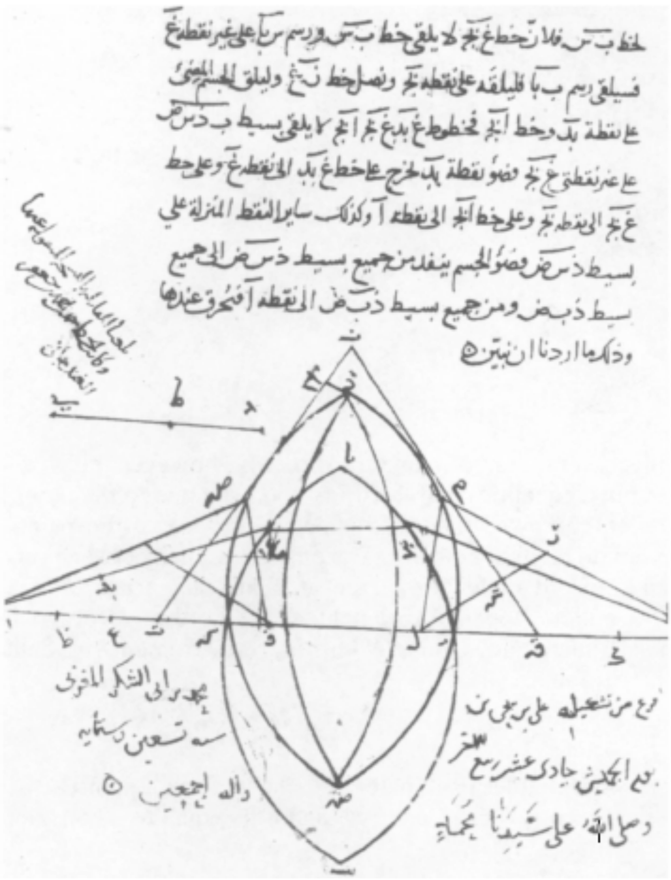

Fig. 2: lentille biconvexe extr. du Traité sur les Instruments Ardents d'Ibn Sahl, Xe siècle . MS 867, folio 26r (Milli Library, Tehran) (Voir: Roshdi Rashed, A Pioneer in Anaclastics. Ibn Sahl on Burning Mirrors and Lenses, ISIS 1990, 81, p. 487)

«Ruisseau gelé»

Ce savoir-faire existait ailleurs dans la culture islamique? La Sérendipité nous fait trouver la réponse dans un roman, «Le Conseil d'Égypte» de l'écrivain sicilien Leonardo Sciascia (1921-1989), qui nous met sur la piste d'Ibn Hamdis.[10]

Le célèbre poète sicilien, Abdaljabbâr Ibn Abî Bakr Ibn Muhammad Ibn Hamdîs (né en 1056 à Noto, près de Syracuse, mort en 1133, à Bougie (Bijâya, Vgayet), Est de l'Algérie, ou à Mallorque, a laissé le poème-énigme que voici:[11]

Ceci est un ruisseau gelé tenu avec la main
Dans lequel le regard plonge en quête des perles de l'esprit
Il revêt de lumières les lignes obscurcies
Comme si une fontaine de lumière jaillissait hors de lui
Il révèle à l'oeil l'écriture du livre
Aussi clairement que l'air, mais son corps est de pierre
Il humecte les joues d'une blessure causée par une sueur
Qui est en lui, reposant sur elles comme une rivière gelée
J'ai passé mes yeux fatigués au collyre de son essence,
L'acuité visuelle n'est-elle pas acquise par le collyre des joyaux?
Il est comme l'esprit d'un homme sagace qui s'en sert
Pour résoudre une énigme rebelle et au déchiffrement ardu

Quel excellent auxiliaire pour un vieillard aux yeux fatigués
Et dont l'âge à ses yeux a rapetissé l'écriture
Grâce à lui il voit grossir les formes des lignes
Comme l'élément de l'eau dans lequel un poil est agrandi.

Il paraît clair que l'objet décrit est une aide visuelle en cristal de roche, car «ruisseau gelé » en est une paraphrase poétique. Ce poème, qui n'est pas une description technique, est le premier récit poétique connu sur l'utilisation d'une lentille grossissante en cristal de roche comme aide à la lecture! Mais de quel objet s'agit-il? Pour certains ce sont des lunettes rudimentaires [12]; prudemment dit, c'est une sorte de loupe.

Induits en erreur par le titre annonçant que ce poème est la description d'un calamus, tous les éditeurs: Celestino Schiaparelli (1897), Ihsan Abbas (1960), et Yusuf Eid (2005), ont donné dans le panneau, et jusqu'à maintenant la signification du poème passe encore inaperçue!

Il offre néanmoins une occasion d'amener les élèves à comparer le style sobre et précis d'al-Biruni à celui poétique d'Ibn Hamdis décrivant des objets similaires, avant d'accéder à un niveau supérieur d'abstraction qu'est la langue des mathématiques comme préparation à la narration de recherche qui combine mathématiques et langage.

Taqiyaddin et son télescope

Selon l'histoire officielle, le télescope fut inventé vers 1608 par Hans Lippershey, un lunetier néerlandais d'origine allemande; peu après la nouvelle de cette formidable découverte, Galilée construisit son propre télescope en 1609 et le dirigea vers le ciel. Le reste, c'est de l'histoire, mais ce n'est pas toute l'histoire...

À la fin de son livre sur l'optique, *Nûr hadaqat al-absâr wa-nawr hadîqat al-anzâr* «La lumière de la pupille des yeux et les fleurs des jardins des regards», achevé en 1574, le célèbre astronome, mathématicien et ingénieur ottoman, Taqiyaddîn ibn Ma'rûf (né vers 1525 à Damas, mort en 1585 à Istanbul) écrit ceci:

«A partir de celà, nous sommes parvenus à fabriquer une lentille (*billawra*) par laquelle nous voyons les choses qui sont cachées en raison de la distance comme les lunules les plus fines et les voiles des bateaux qui se trouvent à des distances excessives et que la vue ne peut saisir avec les yeux les plus perçants; ainsi la lentille qui fut fabriquée par les philosophes grecs qui l'ont placée dans le phare d'Alexandrie. Si Dieu Très Haut me fait la grâce de me donner le temps, je composerai un traité sur sa fabrication et sur la manière de voir grâce à elle, si Dieu Très Haut le veut.»[13]

Ce texte, qui est clairement la description d'un dispositif fonctionnant comme un télescope, antérieur de 34 ans à celui de Galilée, est cependant peu spécifique, car l'auteur annonce la rédaction d'un traité à part. Ceci pourrait être une invitation à discuter en groupe de cet instrument et à en imaginer d'abord les composantes techniques, avant d'en aborder l'aspect optique et mathématique.

Conclusion

Les lentilles grossissantes pour la lecture étaient connues dans le monde islamique dès le Xe siècle. Elles n'apparaissent pas dans un environnement scientifique, à une exception près: résultat inattendu, le télescope est l'aboutissement de l'étude de l'optique arabo-islamique au XVIe siècle. La préhistoire des aides optiques se trouvait donc enfouie dans des sources arabes et des recherches futures menées avec prudence devront clarifier si, quand et par quelles voies ce savoir-faire se serait transmis du monde islamique à l'Europe médievale. Ceci impliquerait une réappréciation des apports de la culture islamique dans ce domaine du savoir. Ces textes constituent donc une nouvelle donne dans l'histoire des sciences, à savoir l'ouverture d'un champ nouveau de connaissance et d'investigation.

Par ailleurs, leur intégration dans un enseignement interdisciplinaire reliant mathématiques, physique et histoire des sciences permettra un enrichissement de l'enseignement des mathématiques (utilisation possible des Fig 1 et Fig 2). Joints à la valorisation des cultures associées à l'immigration, tous ces éléments seraient aussi, en définitive, une pierre dans l'édifice du Vivre ensemble.

À travers des lentilles islamiques (références)

[1] Benfeghoul, Farid, Through Islamic lenses: Rock crystal and others gems as visual aids. In: Seeking transparency (sous presse)

[2] Ahmed Djebbar (en coordination avec Cecile de Hosson et David Jasmin). *Les découvertes en pays d'Islam*. Paris: Éditions le Pommier, 2009.

[3] Abbou Hershkovits, Keren, and Mazor Amir, "Spectacles in the Muslim World: New Evidence from the Mid-Fourteenth Century." *Early Science and Medicine* 18-3 (2013): 291-305.

[4] Rosen, Edward, "The Invention of Eyeglasses," *Journal of the History of Medicine and Allied Sciences* 2 (1956), 13-47; 183-218.

[5] Schmitz, E. H. [La préhistoire de l'invention des lunettes. Ibn al-Haytham eut le premier l'idée d'assister la vision au moyen de lentilles polies], Zeitschrift für Ärztliche Fortbildung 48 (1959), 63-68. (all.)

[6] Hirschberg, Julius, *Geschichte der Augenheilkunde bei den Arabern*, Leipzig 1908, pp. 1-251.

[7] Clément-Mullet, J. J. *Essai sur la minéralogie arabe*. Paris: Imprimerie Impériale 1868, p. 128, 180.

[8] Pline, *Histoire naturelle*. Tome second. Trad. franç. É. Littré. Livre XXXVII, XVI.

[9] al-Bīrūnī, Abū r-Raiḥān Muḥammad b. Aḥmad. *Kitāb al-Jamāhīr fī maʿrifat al-jawāhir*, ed. F. Krenkow, Hyderabad 1355 [1936/27], p. 63-64.

[10] Sciascia, Leonardo. *Le Conseil d'Égypte*. Roman traduit de l'italien [*Il Consiglio d'Egitto*, Turin 1963] par Jacques de Pressac, Paris: Denoël, 1965, p 13-14.

[11] Ibn Ḥamdīs. *Il canzoniere di Ibn Ḥamdîs*. Pubblicato da Celestino Schiaparelli. Roma: Tipografia della Casa Editrice Italiana, 1897. (= Pubblicazioni scientifiche del R. Istituto Orientale in Napoli. Tomo I), (poème n°125), p. 172-173.

[12] Gari, Lutfallah, "The Invention of Spectacles between the East and the West". (http://www.muslimheritage.com/article/invention-spectacles-between-east-and-west) (accessed 28.1.2018)

[13] Rashed, Roshdi. *Géométrie et dioptrique au Xe siècle. Ibn Sahl, al-Quhi et Ibn al-Haytham*. Paris 1993, p. 234.

Salinity study in the river Lima estuary: an interdisciplinary project at secondary level

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Abstract: In this paper we will point out an interdisciplinary project with two classes of 10th graders while they research possible relations among salinity of the river water and three different variables: depth, distance to mouth and temperature. We will present excerpts of three written reports made by one group of each theme showing the ways chosen by the students to accomplish the task while using sensors, Google maps and graphing calculators. Procedures used, results and conclusions were also orally presented by four groups of students in a Mathematical Congress usually held in the school they attend. We draw some conclusions about this type of work and present future perspectives.

Key words: Interdisciplinarity; project; functions; graphing calculator.

Résumé: Dans cet article, nous signalerons un projet interdisciplinaire avec deux classes de 10ème année tout en recherchant des relations possibles entre la salinité de l'eau de la rivière et trois variables différentes: la profondeur, la distance à la bouche et la température. Nous présenterons des extraits de trois rapports écrits réalisés par un groupe de chaque sujet montrant les moyens choisis par les étudiants pour accomplir la tâche en utilisant des capteurs, des cartes Google et des calculatrices graphiques. Les procédures utilisées, les résultats et les conclusions ont également été présentés oralement par quatre groupes d'étudiants à un Congrès Mathématique habituellement tenu à l'école qu'ils fréquentent. Nous tirons quelques conclusions sur ce type de travail et présentons des perspectives futures.

Mots-clés: Interdisciplinarité ; projet; les fonctions; calculatrice graphique.

Background

Nowadays, education for all (UNESCO, 2000) demands variety and diversity of teaching and learning methods. It is necessary to find the best way and the most efficient resources in order that all students learn. In addition, in a world in constant mutation students must be prepared to face unpredictable situations. Thus, it is no more possible to teach mathematics in an abstract style, disconnected from real life (NCTM, 2000).

In this school year, the Portuguese Ministry of Education launched a national flexible curriculum management project that was adopted only by schools that have joined the experience. It aims at promoting better learning that develops higher level skills, centred in schools, their students and teachers, and allowing the management of the curriculum in a flexible and contextualized way, recognizing that the effective autonomy in education only is fully guaranteed if the curriculum is itself autonomous (DGE, 2017). In this scope, the document that defines the essential learning (DGE, 2017) sets that the objectives of acquisition and development of knowledge, skills and attitudes, and their mobilization in mathematical and non-mathematical contexts, are associated with the learning contents of each theme and practices supporting sustainable, comprehensible and transferable learning. This document defends the development, among other things, of problem solving, modelling activities, or projects that mobilize acquired knowledge or foster new learning.

Although the school where I work as a teacher didn´t join this national experience, the Principal encouraged the emergence of such a work creating a school project related to the sea named *Mar Maior*. In the scope of this large project, three teachers having two 10th grade classes in common, Mathematics, Physics and Chemistry as well as Biology and Geology teachers, engendered an interdisciplinary project involving both classes in the research of the salinity in the estuary of the river that crosses and empties into the city. This study aimed to analyse the variation of the salinity with deepness, temperature and distance to mouth.

This presentation refers to an exploratory study about an interdisciplinary project work used by secondary students in the current school year and aims to identify characteristics of the task proposals and/or the work which can foster involvement and endeavour. These students attend a public secondary school in a small town in the north of Portugal and the author of this paper is their Mathematics teacher.

Projects and interdisciplinarity

Freudenthal (1973) showed that students gradually develop mathematical understanding from the real problems of everyday life. According to the creator of the *realistic mathematics* concept, the exploration and solution of real life problems allows students to go through more and more complex levels of mathematical thinking, and, only at an appropriate stage of cognitive and cultural development, to reach abstraction. Also, according to Van den Heuvel-Panhuisen (2001), context problems function as a source for the learning process and real life situations are used both to constitute and to apply mathematical concepts. In a first step, students develop strategies closely connected to the context; Later on, certain aspects of the context can give access to more formal mathematical knowledge.

Interdisciplinary work is based upon transversal and integrative learning articulating several areas, which should stimulate curiosity and will for acquiring knowledge. Students involve in the design, implementation, evaluation and communication of a project. Competencies like organisation, autonomy, cooperation, creativity are fostered. Mathematics and Science must be viewed as related disciplines, describing phenomena by models in order to act and deal with them in a sensible way (Michelsen, 2006). And, according to this author, modelling real-world to help organize the physical world is one of the most common uses of functions.

In the same line, the National Council of Teachers of Mathematics (NCTM, 2000; 2014) states the importance of the use of different functions in mathematical modelling to provide students with efficient means for analysing and describing their world. Technological tools that allow, for example, the construction of graphics and the search of the line of best fit foster the connection with other academic disciplines and develop a deeper comprehension of real life phenomena.

In the last decades, interdisciplinarity research and education are a major trend at national and international levels. Interdisciplinarity is essential to the creation of new knowledge, so the challenge is now to make interdisciplinarity a real force while continuing to build on the strength of the disciplines (LERU, 2016). According to this League of European Research Universities, there are some different degrees of collaboration between disciplines. They are, from low to high: (a) Disciplinarity -Compartmentalisation, shared background, disciplinary community, self-reliance ; (b) Multidisciplinarity - Shared topic, communication, juxtaposition of perspectives, autonomy ; (c) Interdisciplinarity - Integration of disciplinary insights, cooperation, interdependence ; and (d) Transdisciplinarity - Problem-solving, implementation, collaboration between academic institutions and other actors.

We believe, as do the organisers of the subtheme 3 of CIEAEM 70, that it is necessary to work collaboratively to better understand the phenomena that a unique discipline cannot catch and explain completely (2nd Announcement CIEAEM 70, 2018).

In a meta-analysis about the effects of integrative approaches among science, technology, engineering, and mathematics (STEM) subjects on students' learning, Becken and Park (2011) conclude that students who were exposed to integrative approaches demonstrated greater achievement in STEM subjects, and that integrative approaches provide students with a rich learning context to improve student learning and interest. However, teachers in STEM fields lack information on the benefits of integrative approaches.

Methodology

We adopted a qualitative exploratory approach through an instructional strategy involving students in the experimental research work in a common project around the possible variation of salinity in dependence with three different variables. The participants were 47 students of two classes of 10th grade of a secondary school and their three common teachers of Mathematics, Physics and Chemistry, and Biology and Geology. The data were collected in a descriptive and interpretive way including observation and photography of the students' work, notes about the class by the teacher, conversations among the three teachers, analysis of the written reports made by the students and observation of their oral presentation in the Mathematical Congress, and a final questionnaire to the students.

In the beginning of the project the teachers explained to the students the objectives of the project, which was enclosed in the major school project *Mar Maior*, and students were given a script of the work that included the experimental protocol.

In a first phase, with the support to the Port Authority of Viana do Castelo, fluvial routes were carried out with students from both classes of the 10th grade and their respective teachers of Mathematics, Physics and Chemistry, and Biology and Geology. The collection of water in various locations and depths was done with a bottle of Van Dorn. The different samples were bottled and labelled; sae figure 1.

Figure 1. Fluvial routes

The second phase was held in the Chemistry Laboratory. Students, working in groups of four, learned how to work with the salinity and temperature sensors connected with a graphing calculator, and then obtained the data which were relevant to the study assigned to their respective group, according to the distribution previously made by the teacher, from which we show an excerpt (Figure 2).

tue 27/02 **10º A**	**First shift**	**Group 1 (G1T1A)** Salinity reading at three depths in two Places: A - Mouth and B – Eiffel Bridge	**Group 2 (G2T1A)** Salinity reading at the same depth in three Places: A - Mouth, B – Eiffel Bridge and C – New	**Group 3 (G3T1A)** Salinity reading at six temperatures using sea water. **Work**: Data

Figure

		Work: Data from three depths in two places : A and B Use of salinity sensor.	Bridge **Work**: Data from three distances to mouth in three places : A, B and C. Use of salinity sensor and Google map.	from six temperatures. Use of heating blanket, salinity and temperature sensors.

2.

Excerpt of the assignments to the groups

Each group of students then measured the salinity according to one of the variables: depth, distance to mouth and temperature, as shown in Figure 3.

Figure 3. Measuring the salinity in the Chemistry Laboratory

In the third phase, they worked in the mathematics classroom (Figure 4) beginning to learn how to input the data in the graphing calculator. They registered in a table the data obtained by the sensor(s) and when required by the GPS of the boat and the Google map. Then, after an explanation of the subject, they created the cloud of dots and generated a line regression of best fit. In subsequent work the data were analysed and results were recorded.

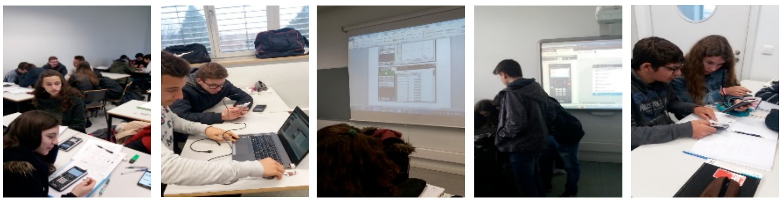


Figure 4. Working in the mathematics classroom

Each group produced a written report of their work.

At the end of the second school term, on the 22nd March, a presentation was made by a selection of nine of these students in the Mathematics Congress (see Figure 5), which is an initiative of our school, that lasts from 2009, where students have space to talk about mathematics in a more informal way, communicating research and discoveries, solving problems, telling mathematical jokes and stories, and challenging their colleagues with puzzles and other problems.

Figure 5. Presentation in the Mathematics Congress

At the end of the work students filled out a survey about the likeness, the commitment, the importance of interdisciplinarity, the favouring of mathematical understanding and possible changes in their perception about the nature of mathematics.

After that, results were used and explored in the different disciplines: study of functions in Mathematics, concentrations and density in Chemistry, and influence of salinity on plant germination in Biology.

Results and discussion

I will present here the results of three groups, as they were communicated in the Mathematical Congress. However, these students were not selected because they had the best results but uniquely because they proposed themselves and accepted to communicate their work to an audience of almost 300 people. The fourth group presented the context, background and aims of the project, adding some photos to illustrate de different phases.

Salinity ant depth

A spread sheet with the depth (p) and the corresponding salinity (s) was built, and then the dots cloud, the linear regression and the analysis of the correlation coefficient. Then a line of best fit was tried, with the logarithm function, as shown in figure 6.

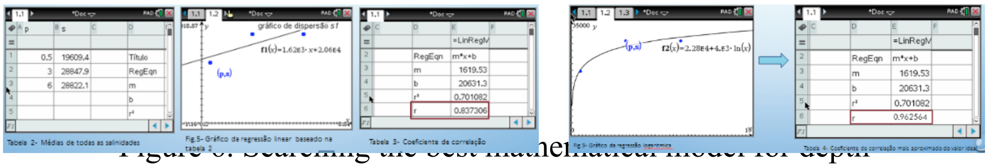


Figure 6. Searching the best mathematical model for depth

The conclusion of this group work was:

From the logarithmic regression graph we conclude that the salinity increases with depth, but not linearly; there is a significant increase in the lowest depths (between 0.5 to 3 meters) and a minimal increase at greater depths (between 3 and 6 meters deep).

However, as this collection was made with the tide rising, it would be necessary to repeat the collection at low tide to effectively compare the results.

Salinity and distance to mouth

This group collected the GPS coordinates furnished by the boat in the trip and then, using Google Map (Figure 7), obtained the distance to mouth from the three points.

Figure 7. Using Google map to calculate kilometre distance to mouth

A spread sheet with the distance to mouth (d) and the corresponding salinity (s) was built, and then the dots cloud, the linear regression and the analysis of the correlation coefficient (Figure 8).

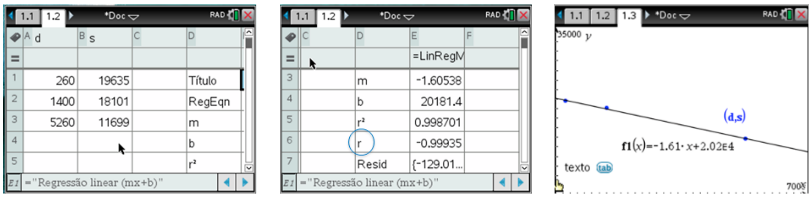


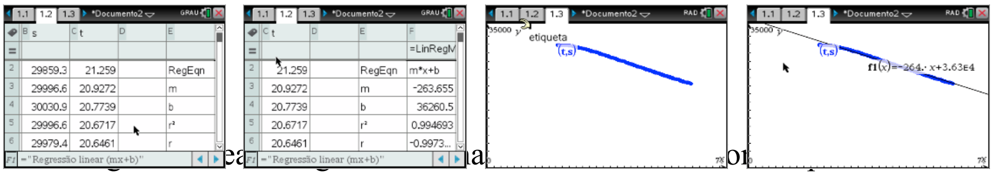
Figure 8. Searching the best mathematical model for distance to mouth

The conclusion of this group work was:

We can then conclude that the greater the distance to the mouth the lower the salinity value, and this variation is almost linear.

Salinity and temperature

This group collected 60 values of temperature and the corresponding salinity. A spread sheet with the temperature (t) and the corresponding salinity (s) was built, and then the dots cloud, the linear regression and the analysis of the correlation coefficient, as shown in Figure 9.



The reported results and conclusions of this group work were:
Correlation coefficient module close to 1; Negative slope; The variation is represented almost perfectly by a line.
With the slope of the line and the coefficient of correlation we can state that:

- *The salinity decreases as the temperature increases;*
- *It is represented in a quasi-linear fashion;*
- *These results could be predicted;*
- *Our Temperature experience has a high number of results, therefore:*
  - There is great precision in the results obtained;
  - We avoided random errors;
  - There may still have been systematic errors.

In synthesis, all the groups presented the results and conclusions, some more complete than others but all revealing effort and interest in the accomplishment of the work they were assigned.

So, and to answer the main question of this study, this project task, having several components in and out of the classroom, including in the exterior of the school, and the contribution of at least three disciplines, proved to be adequate to foster involvement and endeavour.

Concerning the final questionnaire, I will present some quotations, divided in four categories, that reveal the importance that students attributed to this work.

Relating to students' commitment:

- *This work allowed to see Mathematics from another prism and to realize that this is not only about generic problems. It can also be used in research situations and interesting* *projects like this one.*

- *This work provided greater affinity among the students as they had to work together.*
- *It was possible to learn in an environment other than the classroom environment. We had the opportunity to be outdoor in real context.*
- *The work was also important for improving group spirit.*

Relating to the learning of mathematics topics:

- *Since this work has addressed the topic of functions, in addition to the knowledge acquired in the classroom, research was necessary, making it possible to increase the knowledge and understanding of functions.*
- *We used learning in situations different from those we normally use and this was a stimulus to acquire even more knowledge, which, in this case, were the functions.*

Relating to the importance of interdisciplinarity:

- *Interdisciplinarity is very important for learning, since science does not only depend on one area and it is important multidisciplinary knowledge in future professional activities related to the scientific-technological area.*
- *The disciplines are interconnected, as we saw in this matter, and this is a different way of applying our knowledge or even obtaining more. It is also captivating the dynamics created among several topics that we did not think it was possible to relate.*
- *It was interesting to relate Mathematics with Chemistry and Biology.*

Relating to students' perception about the nature of mathematics:

- *This work allowed to see Mathematics from another prism and to realize that it is not only about generic problems. It can also be used in research situations and interesting projects like this one.*
- *This work gave me an idea of how theoretical mathematics can be applied to real-life situations and gave us a greater understanding of how the world works.*
- *This work made me realize that mathematics can be applied in any situation, both in our daily life and in extraordinary situations. It was interesting to be able to apply what we had learned so far in a situation with a completely different context.*
- *Thus, we have proof that mathematics is not only learning how to make computations on paper and discover sides of triangles. On the contrary, we discovered that it is much more than that: it is something that we encounter daily and we apply naturally!*
- *Until now Mathematics had always been worked in the classroom with normal classes and exercises, but this has all changed with this work.*

As we can see, students valued the possibility of working in group; considered that the experimental work and results helped to understand mathematics topics; stressed the importance of interdisciplinarity in the comprehension of the world; and, in a very emphatic way, showed a shift in their perception about the nature of Mathematics.

This work had its own value in the acquisition, by the students, of competences for collaborative work and research experience, including collecting and treating data, learning how to organise the research report, in which drawing cautious and sustained conclusions, as well as presenting publicly their work.

Moreover, the results of this work served as a departing point to the study of some topics in each discipline. For instance, in Mathematics, the teacher used the values obtained to introduce concepts in the study of functions like function, forms of representation, tables and graphics, variable, domain, injective and surjective functions, etc.

Concluding remarks

This work format in which students make experiences, register results and produce conclusions with a certain independence has proved to be valuable in order to foster involvement, endeavour, autonomy, organisation and a team spirit. The joint approach to a real problem shows students the connections among the different scientific disciplines and in what measure each discipline can contribute to the others and to the solution of a problem, giving students a more holistic view of the articulation of knowledge. In addition, some topics of the disciplines curricula may be approached in a different, more real and innovative way (Van den Heuvel-Panhuisen, 2001).

In particular in Mathematics, where Portuguese school curricula are at this time extremely abstract, with a level of rigour and deductive reasoning much above average students characteristics and interests, this type of project work may inspire a new interest in the study of mathematics, as it motivates the introduction of formal mathematics topics (NCTM, 2000). Actually, the very concept of function acted in this case as an appropriate tool for mathematizing relationships between physical magnitudes (Michelsen, 2006). Although we recognize that most mathematic subjects require individual study and training, mathematics must not be reduced to those procedures. It must involve the interpretation of real life situations, reasoning, modelling and communicating. And in this sense the graphing calculator – which is required at this level but is often underexploited - had in this case a key role in modelling and interdisciplinarity.

We also want to stress the importance of communicating and spreading results from an experience, which had in the existence of the Mathematical Congress a positive opportunity of effectiveness.

The students' evidence collected with this work make us feel we took a step to follow the national flexible curriculum management ongoing project in our country.

Of course this collaborative work is very demanding and is thus not possible without the motivation and will of the respective teachers, which is consistent with Becken & Parks (2011) who state that teachers lack information on the benefits of integrative approaches and these depend upon teachers' committment to the integration. In our case, the success is partly a consequence of the good relationship among teachers. These results point out some future collaboration among the three disciplines, keeping the right balance between disciplinarity and interdisciplinarity (LERU, 2016).

References

Becker, K. & Park, K. (2011). Effects of integrative approaches among science, technology, engineering, and mathematics (STEM) subjects on students' learning: A preliminary meta-analysis. *Journal of STEM Education, 12 (5-6),* 23-36.

CIEAEM 70 (2018). Retrieved from https://cieaem70.sciencesconf.org/resource/page/id/8

DGE (2017). *Autonomia e flexibilidade curricular.* Retrieved from http://www.dge.mec.pt/autonomia-e-flexibilidade-curricular#

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: D. Reidel Publishing Company.

LERU (2016). *Interdisciplinarity and the 21st century research-intensive university*. Retrieved from https://www.leru.org/files/Interdisciplinarity-and-the-21st-Century-Research-Intensive-University-Full-paper.pdf

Michelsen, C. (2006). Functions: a modelling tool in mathematics and science. *ZDM, 38(3)*, 269-280.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston: NCTM.

NCTM (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, NCTM.

UNESCO (2000). *The Dakar Framework for Action: Education for All - Meeting our Collective Commitments*. World Education Forum, Dakar, UNESCO.

Van den Heuvel-Panhuisen, Marja (2001). Realistic Mathematics Education as work in progress. In F. L. Lin (Ed.), *Common Sense in Mathematics Education, 1-43. Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education*. Taipei, Taiwan, 19 – 23 November 2001.