INTEGRATION OF CAS IN THE DIDACTICS OF DIFFERENTIAL EQUATIONS

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Abstract: In this paper are described some features of the intensive use of math software, primarily Derive, in the context of modeling in an introductory university course in differential equations. Different aspects are detailed: changes in the curriculum that includes not only course contents, but also the sequence of introduction to various topics and methodologies. For example, the use of software at the beginning of the course follows a black box strategy, in order to emphasis mathematical modeling and the discussion of theory's central problems. Also covered is the enrichment of discussion about classic problems; using CAS potentialities to demonstrate properties without the necessities of investing excessive time in technical details, avoiding long and boring calculations; design and redesign of materials that attempt to guarantee success of information technology integration; combined use of several software packages, and finally, feedback from students and assessment. In addition, examples are presented to demonstrate the support materials utility for high school Calculus.

1. Information technology as central element in the scientific work

Rita Colwell, director of the NSF (National Science Foundation) of the United States, has written (Colwell, 2000) about the unifier role that information technology is playing in research in different sciences due to it allows to tie among different fields of the knowledge, and she states that no field of research will be immune to the explosion of information and information technology, however, she makes another statement that we consider very important that it is when it points that until recent years science had two components, theory and experimentation, but today it has a third component «computer simulation, which links the other two» (p.16). Colwell point out that nowadays scientific matters have grown in complexity as much as in Interdependence and it points out the specific case of the complex mathematical models that are used in biology and in social sciences, she states that many scientific achievements will be reached as far as there exists advances in information technology, «we need this computing power to put it all together: to process the volumes of data, to visualize results, and to collaborate» (p.17).

2. Mathematical modeling and Information technology

In different sectors appears the perception of a fundamental change in mathematical modeling. Indeed, it's possible to say that before the information technology birth, mathematical modeling of a physical process, could be described by the outline of the left in the following figure. Starting from the considerations of Bricio (1992) in his description of the mathematical method, the previous outline could be modify in the way that is shown in the right of the following figure.

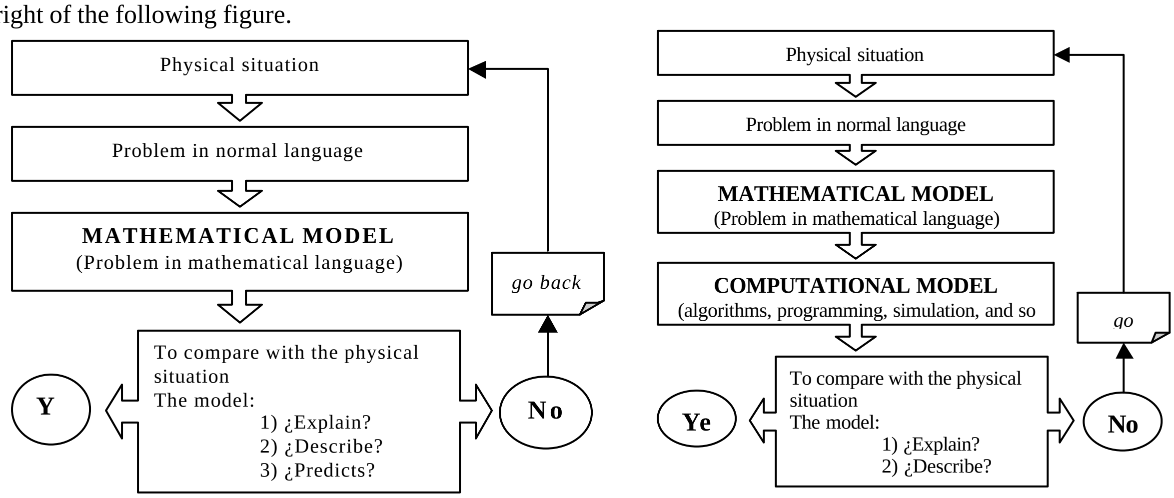

Figure ***Errore. L'argomento parametro è sconosciuto.****: The outline of the mathematical modeling*

Bricio states «that's the way of work at present time in sciences and engineering, and is what constitutes the mathematical method» (p.68). If we accept this point of view it means that prepare mathematically future generations for science and for engineering it implies the formation of professionals skills in this new phase of mathematical modeling.

In this panorama Spunde (1999) points out an interesting question, Will new models attempt discrete descriptions of reality keeping in mind both the physical processes and the computational environment in which the problem is to be solved, by-passing the continuous ideal approximation completely?, that means, will we put on discussion the continuous models that we are so used to?

3. Differential equations and mathematical modeling

Theory of differential equations arose, almost contemporarily with Calculus, at the end of the SXVII, Aleksandrov (1956) aims that this theory inaugurates a third cultural period in mathematics, what allows to stand out the weight that this theory has in the historical and conceptual development of mathematics.

From the beginning, it has been and it continues being an important field of theoretical research and practical applications, it constitutes an extensive and very important branch of modern mathematics and it's a powerful tool for the investigation of many natural phenomena, what determines their strong relationship with mathematical modeling.

As it is known, the first central problem of the theory refers to the origin of differential equations. Before information technology birth, the process of mathematical modeling of a physical situation (see outline presented in **Errore. L'argomento parametro è sconosciuto.**) could be described in a more particularized way by means of the following stages:

1) Description of the physical situation which is under analysis;
2) Construction of the problem in common language;
3) Construction of the problem in mathematical language (construction of the mathematical model);
4) Solution of the problem. (In case that model is an equation, solution of the equation);
5) Analysis and interpretation of the solution.
6) Validation of the model confronting it with the physical situation analyzed to decide if the model is adequate on the required accuracy or if it is necessary to adjust it, if the answer is positive, the problem is solved, if the answer is negative model should be adjusted.
7) In the case adjusting the model and repeat the process with the new obtained model. Some times a model works well during a determined time but in the base of new requirements or new discoveries about the physical situation it necessary to construct a new model.
8) Implementation of the model.

Naturally, this is not the only proposal to describe the stages of the modeling process. For example; a slightly different proposal that describes 7 stages can be seen in Dreyer (1993) and, as he affirms as well, in the case of a specific problem all the stages are not explicitly evident since some of them can be trivial or they have been previously solved, and so on. In the peculiar case of the mathematical modeling with differential equations, stages 1, 2 and 7 are identical. The other stages can be described in the following way:

3) Construction of a differential equation or of a system of differential equations;
4) Solution of the differential equation or solution of the system of differential equations;
5) Analysis and study of the function or of the obtained functions as well as interpretation of their properties with regard to the physical process that is being studied;
6) Determine if the function or the obtained functions describe the process appropriately with the required accuracy. If the answer is positive, go to stage 8;
8) Use the function or the obtained functions to make predictions, in which case care should be taken in determining the time interval for which the predictions are valid.

In agreement to that pointed out by Colwell earlier, the theory of the differential equations could not be outside the reciprocal influence with information technology.

4. Some problems of differential equations theory

Petrovski (1956) generally outlines the problems with differential equations theory (which from now on will simply be referred to as the theory) in a time that information technology was still not largely available and therefore we can consider that it is a question of intrinsic problems of the theory. Since these problems are many and varied, it presents those that he considers more important and at the same time it omits considerations on many other branches of the theory that appear when studying more particular problems or that require a much deeper mathematical knowledge. Although he does not explicitly present a special section about the modeling, he recurrently make references to the bound problems when using these types of equations, illustrating all the problems that appear in each of the stages of the modeling process, at times making some general considerations at others giving concrete examples. In general, we believe that teaching any branch of mathematics should reflect a global vision of the theory that is being studied with the natural limitations of the students age and level being considered. In particular, the teaching of differential equations should punctually reflect each and every one of the central problems of the theory. However, in most cases it seems not to be this way.

A brief hypothetical exercise will be given to analyze the effects of the lack of some of these problems. Considering an extreme case: The total lack of one of them or an important part that will allow the illumination of didactic problems in the traditional courses.

1. An appropriate amount of modeling. Indeed, how do we conceive a course of DE without linking it to the modeling of physical situations? In the extreme case, it would be a course in which exclusively teaches how to solve equations without mentioning neither the origin of the equations nor the utility of having obtained the solution. In a course of this type, the students would be helpless to be able to give a physical interpretation of the solution. They would also very likely retain a mistaken image of the matter: A series of methods to obtain solutions of equations that would not have any relationship with the real world.
2. Some examples with the complete process of modeling. Although the construction process is very difficult,

it is possible always to present some interesting models as well as to teach the students to build some simple models, mainly those that are directly linked with the area of their study. In many occasions, traditional courses never present the complete process. In the best case scenario, training is provided to outline the relative equations to some simple models and the solution is obtained, yet a detailed analysis of the solution is not made nor interpreted or validated with regard to the phenomenon that is to be "studied". As in the previous example, it is very difficult to appreciate the utility of the theory when all the problems terminate once the solution of the differential equation is obtained.

3. An appropriate balance among the three focuses for the solution of an equation or of a system of equations. Generally, traditional introductory courses only follow an analytic approach, ignoring the importance of the other two almost completely. In the best case scenario, a numeric method is touched upon and qualitative aspects are practically never considered. This gives the students a completely distorted vision of the theory as key parts are left out.
4. Analysis and interpretation of the solutions whenever possible. In traditional courses, the problems of differential equations generally finish when one obtains the exact solution. Once obtained, the students continue to the next exercise, the solution is obtained, and then another exercise and so on. In the best of cases the graph of a particular solution is asked and this is only in the case that the graph is not too complex. It is interesting that Petrovski points out that studying the solutions is the *basic problem of the theory*, and not an insignificant problem. It is not possible to "have seen applications" without having studied the behavior of the solutions.
5. Validation of the pattern whenever it is possible. Even in the case of not having constructed a model with the students, it is always possible to present the context in which the model arises and therefore to compare the obtained solution with the referenced phenomenon, carry out small simulations, to make an analysis of the parameters, and so on. In traditional courses if those solutions are not even studied its impossible to carry out the validation process.
6. Some examples of model adjustment. This important part of the modeling process can always be illustrated with some examples and discussed with others. In traditional courses this part is limited by the results of such an activity, the new function or the new obtained functions are much too complex to be analyzed, studied and interpreted that, in the majority of cases, the professor decides not to illustrate this part of the process.
7. Some examples of predictions. Once a differential equation or a system of DE has been solved, this solution should be used to make some predictions. This is a central part of illustrating the problem of the relativity of the model when the time interval for which the predictions are valid. It is best if the predictions are made with models that come from situations with real data as this will provide evidence to confirm or deny the relevancy of the prediction. In traditional courses, this part of the process is found in some examples yet it would seem that it is not given adequate importance.

5. Some problems of the differential equations theory in the Information technology era

The didactic problems evidenced previously can be part of an ordinary introductory course in differential equations and now a new equilibrium is possible if we use information technology to illustrate the said problems in order to be able to introduce the students to a much more complete vision and integration of the theory. Indeed, for some years there is a growing number of books on differential equations which integrate the software use in their study proposals. The philosophy behind these proposals can be found for example in Malek-Madani (1998), Coombes et al. (1995), Gray, Mezzino & Pinsky (1997) and Blanchard, Devaney & Hall (1998). Of these four books, the first one is based on the use of *Mathematica* and *Matlab*, the second one and the third make reference to *Mathematica*, although in the second an edition based on Maple is also presented. The last of them presents general proposals that can be utilized with the appropriate software. As Coombes et al. point out (p. iii) «Traditional introductory courses in ODE have concentrated on teaching a repertoire of techniques for finding formula solutions of various classes of differential equations. Typically, the result was rote application by students of such fundamental aspects of the subject as stability, asymptotics, dependence on parameters, and numerical methods». It is possible to find this same type of critique in the other referenced books or in some articles where experiences of information technology use have been reported in courses on differential equations, such as Evans (1995) working with Derive or in Shay (1997) using the TI-92.

6. Some proposals

Our work experience has been carried out in a technical university in the center of Mexico during the last 5 years with students in introductory courses on differential equations. The vast majority of students are 19-year-olds although older students exist (usually working students). In general, the introductory course is the only one that many students will follow and, therefore, the vision that they have of the theory will be the consequence of this one course. They have worked with *Derive* and with additional software such as *Phaser* or *Cyclone99*, following different introduction strategies of the mathematical modeling. For example, it is possible to take modeling as a conductive axis introducing early most of the problems described in section 3 using the software in a black-box strategy that allows students to concentrate on stages 1,2,3,5,6,7 and 8 using the utilities of *Derive* to solve the necessary differential equations without caring if the equation is of first or second order. In traditional courses this is not possible since, typically, the students are taught firstly solution methods for equations of first order which introduces a limit in modeling since they only present models which conduct equations of this type. With this new outline it is possible to discuss with students the Malthusian model and immediately the logistic model without having to wait for the solutions method. Continuing with this line of thinking, you may immediately discuss predator-prey model without caring that it does not conduct to a system of linear differential equations or in the case of the mass-spring systems discuss from the beginning forced systems and not only unforced systems. This strategy allows emphasis some central questions. Indeed, when delegating to a machine the solution of the differential equation the solution can be obtained and make a detailed analysis of it in such a way that when student studies the solution methods he/she has a clear idea of which are their utilities and limitations. Next, two examples are briefly described of how the software has been used to illuminate several of the stages in the modeling process.

7. Example 1: Mass-Spring systems

In traditional courses, mass-spring systems are presented as an application of second order equations. Due to this situation, these kind of problems should be analyzed after having studied the solution methods of first and second order equations. Typically, applications approach conduct to equations of the form: $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$, and, more infrequently to equations of this type: $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=f(t)$, because in the last case it is necessary to have studied previously solutions methods for non-homogeneous second order equations with constant coefficients. However, with the use of CA type software, several of the stages of mathematical modeling with these kind of problems can be illustrated. In our case the use of *Derive* with a black-box strategy allows us to present the solution of the equation and to work with it. For example:

1. Making a detailed description of the physical situation and of the importance of the analysis in order to later present it to the students in normal language, to evidence clearly all the used assumptions, to translate the problem in mathematical language and to build the model (stages 1,2 and 3);
2. Stage 4 is postponed for later, once focus is on solution methods. However, solution is not given in dogmatic form but rather it is proven with the students that the proposed function is the solution of the equation so that student are convinced of this fact;
3. Once the solution has been obtained, it is worked with to deepen the analysis of the physical situation, to validate and adjust the model (Stages 5,6 and 7). For example, embarking in the traditional way, with a undamped harmonic oscillator, the model is adjusted in order to consider damping, distinct possibilities are analyzed of these kind of oscillators and finally the model is readjusted to consider external forces.

This form of working is liberating for the staff. In traditional courses emphasis is put almost exclusively in stage 4 and until before the large access to CAS was available alternative emphasis were almost impossible. As Coombes et al. (1995) indicate, much of the differential equations ideas are difficult to teach if a computer is not used. Giving priority on the conceptual part of the modeling presents us with two obstacles: In order to have a solution it is necessary to solve the equations with a previously studied method and secondly in order to analyze the solution sophisticated geometric analysis instruments are required.

These are some examples of the problems that are to be solved with the students in some laboratories (it should be cleared up that students have not seen any solution method neither have they seen a formal classification of these kind of equations. Everything is done, for the moment, in an informal and intuitive way).

1. Solve the differential equation $mx''+bx'+kx=0$ using some *Derive* utility file

Upon entering, student should already recognize the equation as a second order equation and he/she should know that *Derive*'s utility DSOLVE2 can be used for which the following result is obtained:

$$x(t)=c2\ \hat{e}^{-t\ \sqrt{(b^2-4\ k\ m)}/(2\ m)-b\ t/(2\ m)}+c1\ \hat{e}^{t\ \sqrt{(b^2-4\ k\ m)}/(2\ m)-b\ t/(2\ m)}$$

This is not the usual form that would be found in the books where the classification is made in terms of overdamped, critically damped and underdamped cases with forms are given by:

$$x(t)=Ae^{\alpha t}\sin(\beta t+\varphi) \qquad x(t)=(C_1+C_2t)e^{\alpha t} \quad x(t)=C_1e^{\alpha t}+C_2e^{\beta t}$$

Respectively. The form provided by *Derive* is similar only to the former. In this case it is requested that students do a second exercise:

2. Proof that under certain conditions the function $x(t)=e^{\alpha t}(C_1\cos\beta t+C_2\mathrm{sen}\,\beta t)$ is also the solution for the previous equation.

The development with *Derive* that a student should carry out is shown in the following figure:

$$\#1:\ \hat{e}^{\alpha \cdot t} \cdot (C1 \cdot COS(\beta \cdot t)+C2 \cdot SIN(\beta \cdot t))$$

$$\#2:\ m \cdot \left(\frac{d}{dt}\right)^2 x + b \cdot \frac{d}{dt} x + k \cdot x$$

$$\#3:\ m \cdot \left(\frac{d}{dt}\right)^2 (\hat{e}^{\alpha \cdot t} \cdot (C1 \cdot COS(\beta \cdot t)+C2 \cdot SIN(\beta \cdot t)))+b \cdot \frac{d}{dt} (\hat{e}^{\alpha \cdot t} \cdot (C1 \cdot COS(\beta \cdot t)+C2 \cdot SIN(\beta \cdot t)))+k \cdot (\hat{e}^{\alpha \cdot t} \cdot (C1 \cdot COS(\beta \cdot t)+C2 \cdot SIN(\beta \cdot t)))$$

$$\#4:\ \hat{e}^{\alpha \cdot t} \cdot ((C1 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)+C2 \cdot (2 \cdot \alpha \cdot \beta \cdot m+b \cdot \beta)) \cdot COS(\beta \cdot t)+(C2 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)-C1 \cdot \beta \cdot (2 \cdot \alpha \cdot m+b)) \cdot SIN(\beta \cdot t))$$

$$\#5:\ (C1 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)+C2 \cdot (2 \cdot \alpha \cdot \beta \cdot m+b \cdot \beta)) \cdot COS(\beta \cdot t)+(C2 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)-C1 \cdot \beta \cdot (2 \cdot \alpha \cdot m+b)) \cdot SIN(\beta \cdot t)$$

$$\#6:\ C1 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)+C2 \cdot (2 \cdot \alpha \cdot \beta \cdot m+b \cdot \beta)=0$$

$$\#7:\ C2 \cdot (\alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k)-C1 \cdot \beta \cdot (2 \cdot \alpha \cdot m+b)=0$$

$$\#8:\ \alpha^2 \cdot m+\alpha \cdot b-\beta^2 \cdot m+k=0$$

$$\#9:\ 2 \cdot \alpha \cdot m+b=0$$

$$\#10:\ \alpha=-\frac{0.5 \cdot b}{m}$$

$$\#11:\ \left(-\frac{0.5 \cdot b}{m}\right)^2 \cdot m+\left(-\frac{0.5 \cdot b}{m}\right) \cdot b-\beta^2 \cdot m+k=0$$

$$\#12:\ \beta=\frac{0.5 \cdot \sqrt{(4 \cdot k \cdot m-b^2)}}{m}$$

$$\#13:\ \beta=-\frac{0.5 \cdot \sqrt{(4 \cdot k \cdot m-b^2)}}{m}$$

1. In the expression #1 the rule of correspondence of the given function is introduced;
2. In #2 the left side of the differential equation is introduced;
3. In #3 the function is substituted in the differential equation and is simplified (#4);
4. Make the coefficient of $e^{\alpha t}$ zero keeping a linear combination of **sin(β*t*)** and **cos(β*t*)** (#5);
5. Make the coefficient of **sin(β*t*)** and **cos(β*t*)** zero (expressions #6 and #7, respectively);
6. Make the coefficient of $\mathbf{C_1}$ and $\mathbf{C_2}$ zero (expressions #8 and #9, respectively);
7. Finally, solve the system (expressions #10 to #13) to reach the conclusion that the given function is the solution of the differential equation only for $b^2<4mk$.

The same is done in other situations. Afterwards ask the students to solve problems of the type:

3. **Given the form of the solution $x(t)=Ae^{\alpha t}\sin(\beta t+\varphi)$, write it in terms of parameters and initial conditions.**

The students should arrive to the following:

$$x(t)=\left(2\sqrt{m}\sqrt{\frac{mv_0^2+bx_0v_0+kx_0^2}{4mk-b^2}}\right)e^{-bt/(2m)}\ \mathrm{sen}\left(\frac{\sqrt{4mk-b^2}}{2m}t+\arctan\left(\frac{x_0\sqrt{4mk-b^2}}{2mv_0+bx_0}\right)\right)$$

In paper and pencil environment this is unmanageable in a normal situation but with this forma we have the advantage that permits us to simulate as much with the parameters as with the initial conditions. In effect, using this form asks the student to simulate with each of the parameters and with both initial conditions and that he/she describes the effect of each of them in the form of the solution. In the case of forced vibrations of the type: $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\gamma t$, the general solution is in the form

$$x(t)=Ae^{\alpha t}\sin(\beta t+\varphi)+F_0M(\gamma)\sin(\gamma t+\theta)$$

with $\alpha=-\frac{b}{2m}$, $\beta=\frac{\sqrt{4mk-b^2}}{2m}$, $M(\gamma)=\frac{1}{\sqrt{(k-m\gamma^2)^2+b^2\gamma^2}}$, $\theta=\mathrm{atan}\left(\frac{k-m\gamma^2}{b\gamma}\right)$

The values of **A** and φ can be determined starting from the initial conditions but contrary to other cases previously analyzed with the students, in the general case the expressions remain extremely complex. For example, the most simple of them is φ which follows (here in *Derive* we write *f* instead of γ)

$$\varphi=-ATAN\left(\frac{\sqrt{(4\ k\ m-b^2)}\ ("F0"\ (f^2\ m-k)+"x(0)"\ (b^2\ f^2+(f^2\ m-k)^2))}{"F0"\ b\ (f^2\ m+k)-(2\ "x'(0)"\ m+"x(0)"\ b)\ (b^2\ f^2+(f^2\ m-k)^2)}\right)$$

Now it is extremely complex to obtain the rule of $x(t)$ only in terms of the parameters of the system, the initial conditions and the forced term. It fails to illustrate generally what happens in these cases as well as to evidence some of their properties

8. Example 2: Population Models

Now, in some cases, the mathematical modeling is an integral part from the curricular nucleus of many university level courses. Yet, also, at high school level as it is pointed out in Oldknow (1997). Part of the "taste" of mathematical modeling with differential equations can be introduced in courses of Calculus at high school level if we use software in stage 4 with a black-box strategy that allows students to concentrate on the study and analysis of a function of one variable. For example, in an experiment that the Author carried out with Italian high-school science students during several sessions in February of 2000 (18-19 year-old students) the Malthusian and logistic models were analyzed with real data relative to the world population in such a way that they were who decided about the validity of the models. In this case we followed the same strategies and we used

However, that does not prevent us to consider diverse particular cases that allow us to illustrate phenomena so important as resonance or beating.
In effect, with *Derive* we can illustrate the resonance curves making **i** seen that the system enters in resonance in the way that $b \to 0$ and $\gamma \to \omega = \sqrt{\frac{k}{m}}$ as shown in the side figure.

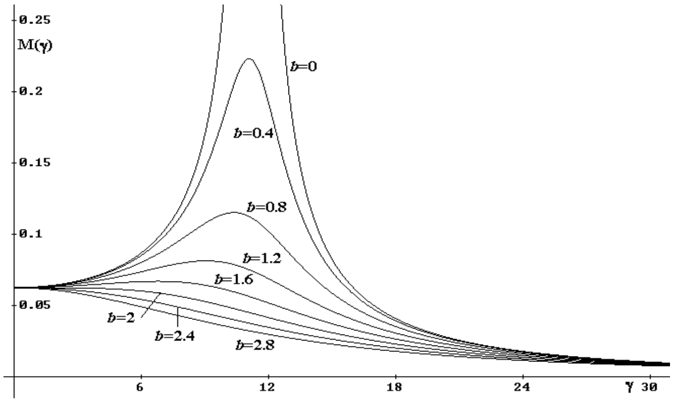

Figure ***Errore. L'argomento parametro è sconosciuto.****: Resonance curves*

the same materials that we uses with Mexican university students without finding significant differences between. the work of the Italian high school students and the Mexican university students. In the first group, the management of *Derive* was elemental while in the second group it was never used before.
Working with this type of models has the advantage that their speaks about realties to the students and it allows the illumination of some other types of interesting phenomena when working with mathematics software.
For example, in the case of the Italian students, working with the Malthusian model $\mathbf{P(t)=3340e^{0.02t}}$ students are asked to calculate the earth population in the years 2100, 2200 and 2600, obtaining the following results: $\mathbf{P_{2100}=49{,}698}$ millions, $\mathbf{P_{2200}=367{,}223}$ millions and $\mathbf{P_{2600}=1{,}094}$ billions! When asked the question "Do these values have significance for you?" 100% of the students answered no and that they did not know if these values are probable or not because they are not accustomed in their mathematics courses to compare their results against real data and this prevents them from evaluating the validity of the model (here it must be noted that according to UN, the population of earth in the year 2000 was 6,000 million).
So they are not accustomed to confront the data obtained with the reality that of 20 students, only 4 wrote correctly the function $\mathbf{P(t)=3340e^{0.02t}}$ and all others made errors that caused them to conclude $\mathbf{P_{1925}}=\frac{\mathbf{3340}}{e^{4/5}}$ people! They did not consider that the result was not a whole number, so the calculations continued concluding that $\mathbf{P_{1950}}=\frac{\mathbf{3340}}{e^{3/10}}$ people, an so on.
Even more astounding is the fact that 6 students wrote $p=3340e^{0.02}t$ and obtained that $\mathbf{P_{1925}}= -133600e^{1/50}$ people!!! Furthermore, 2 more wrote $p=3340e0.02t$ and obtained $\mathbf{P_{1925}}= -2672e$ people! In all the cases, the students continued working as if nothing was in error in spite of having obtained negative or non-whole numbers for the population.
It is very probable that this is the result of lack of work in the schools with real data. In this sense, software can make a lot of sense for the students as it allows them to take these kind of problems to their classes and, therefore, may cause the students to reflect upon the proposed analysis. This situation is a regular occurrence in many traditional differential equation courses when, for example, electric circuit applications are worked and the values assigned to the resistance R, to the inductance L, and to the capacitance C is whole, as in the case when they are outside the range of possible values of these physical magnitudes that according to Blanchard, Devaney & Hall (1998) a typical, off-the-shelf circuit might have parameter values R=2000 ohms, $C=2\times 10^{-7}$ farads and L=1.5 henrys.
This type of situation illuminates another interesting situation when one works with software: Blind faith in the results for what is needed to design didactical pathways to counteract it like is suggested in Zhao (1998).
In the case of logistic model it is interesting to point out that it allows discussion of problems which otherwise would be hidden.
In Braun (1983, p. 31) is presented a formula which Pearl & Reed introduced in order to analyze the U.S. population «Using the census of 1790, 1850 and 1910 it was found that $\boldsymbol{a}$**=0.03134** and $\boldsymbol{b}=\mathbf{1.5887\times 10^{10}}$» pointing out later that "These results are surprising" to predict the American population. In Braun's book, prediction values are presented and surprisingly agree with the real data up to 1950 leaving the reader with the impression that the model is very accurate but when it is applied to later decades with the same parameter values, the error ranges from 1.1% to 11.2% to 17.4% to 22.8% until finally arriving at 27% in 1990. In Nagle & Saff (1982) the census of 1790, 1840 and 1890 are used and they improve the predictions, yet already in 1980 the error begins at 7.1% and grows to 12% in 1990. Likewise in Braun's book a table is presented that gives the same impression. With respect to this, an interesting question arises: It is true that the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ are determined by 3

censuses with the condition $\boldsymbol{t_1 - t_0 = t_2 - t_1}$, yet when choosing diverse periods different values are obtained and even negative parameter values which do not make any sense.
Once again, illuminating this type of problems with the students allows the illustration of the validity relativity of the mathematical model.

9. Students Opinions

In these types of experiments, at the end of the course the students are asked their opinions on four simple, open questions: What is your opinion about the content? Methodology? Evaluation? And can you give some additional comments? Naturally, these questions are not part of any research. However, they help explore the students opinions in an informal way and allow the students to answer openly. It is very interesting to know the answers of the students as they comment upon the class as much as the professors and researchers and may show a personal reflection of the students about their own actions. These comments, in the future, can become a valuable asset to redesign integration procedures of integration information technology in mathematical education. Below a brief selections of opinions are presented that comment upon the previously discussed aspects (The original writing is respected, the cursive writing is ours).

1. «I saw a lot of applications in the real life, because *always teachers had taught us mathematics without any real application*. For the first time in my life I really could applied contents in real things. *I truly studied much more* because applying software on more complex problems was used which was more close to real situation» [Rocio, 00];
2. «It was very interesting to know the mathematical meanings, how it works and how they are applied. *I used to believe that mathematics did not have any use* yet the applications were truly interesting as they applied to real life» [Julio Cesar, 00];
3. «With the software use I believe it helped us increase our capacity to understand the concepts and applications» [Damaris, 00].

Here are some response to the question "Do you believe that with the software use you learned mathematics more or less?" Some of the students responded:

4. «I believe that I learned more as we were devoted more to the applications and mathematical reasoning in each topic, and the assistance that the software gave us saved us time, time that we could use in more essential things than solving algorithms by hand» [Claudia, 00];
5. «I learned more, mainly in understanding because in the last courses, I passed, I had a good grade and everything, but I never had a good knowledge of why we applied this formula, this method, and so on. However, the software gives you a more clear vision of what you are doing (for example when do you make graphs)» [Moises, 00].

10. Final remarks

As has been pointed out in several occasions, the software use helps to develop the students abilities in modeling without having to introduce enormous calculations and permits us to enrich the solution (Oldknow, 1997) or to investigate more complex models than normal (Mitic & Thomas, 1994). Or, as Böhm describes (1994) the textbooks only present simple standard models due to the great numbers of calculations to be made. «The existence, versatility and power of technology make possible and necessary to re-examine what the students should learn of mathematics as the form in which they should make it» (NCTM, 2000, p. 25). Now it is possible to present in class a richer and more interesting examples tied to the applications in order to illustrate concepts and techniques, which not only enriches the class yet also the homework and exams. Now it is possible to achieve a new balance among the different focuses used to study a differential equations: numerical, algebraic and qualitative. On the other hand, information technology allows us a better conceptual historical-genetic approach to the theory of differential equations, to introduce the students to the process of mathematical modeling in a more complete way and to generate in students contemporary skills and methodologies which will be useful to them in the future.

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