ONE GEOMETRICAL SITUATION AND THREE COMPETITIONS PROBLEMS

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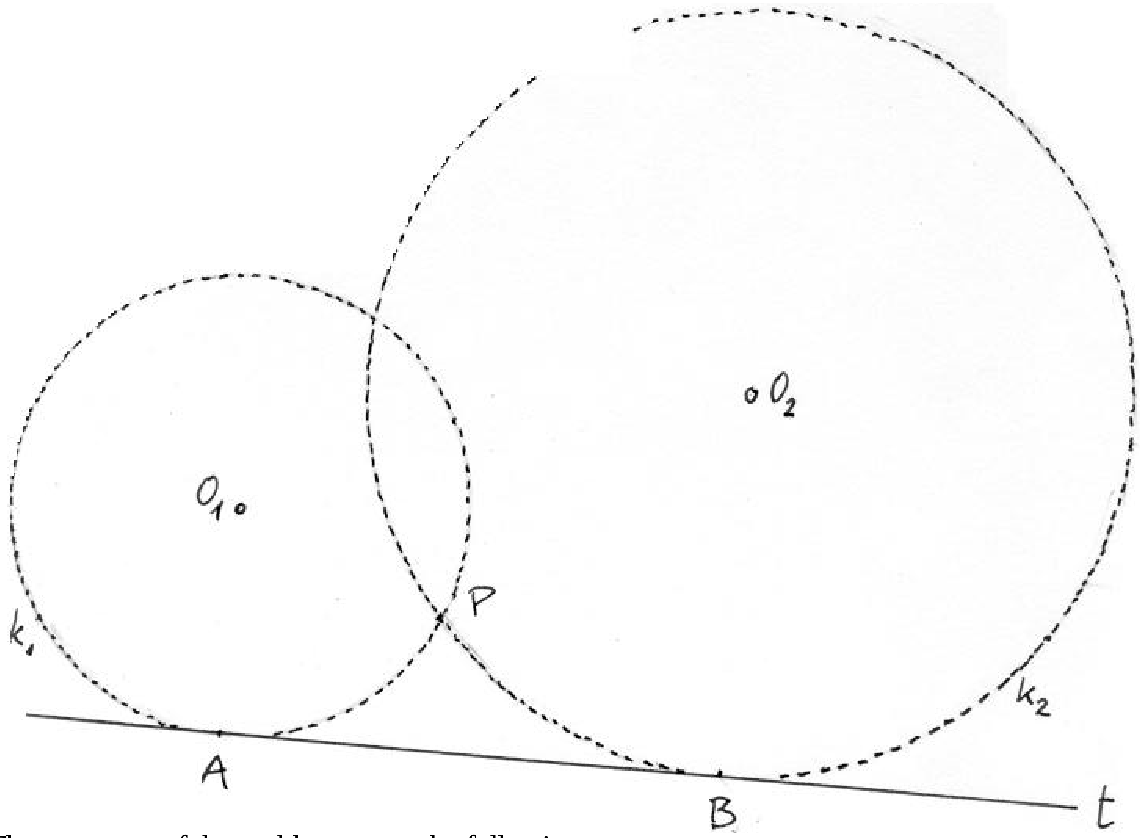
In march 1999, july 2000 and september 2000, a without precedent situation takes place in the Asian Pacific Mathematics Olympiad, the International Mathematical Olympiad and the Iberoamerican mathematical Olympiad, respectively : One problem of each one of these competitions repeat the same geometrical situation.

That a same geometrical situation produces different problems is, of course, not new : all the problem-solvers know several examples of this. But that in three competitions, very close in the time, three problems of these characteristics were proposed, is a without doubt exceptional situation. Because the first work of the Jury's members is precisely to avoid similar problems, for the sake of the fair play. And althought the Jurys were different, probably they have "no empty intersection", at least in the I.M.O. and in the Iberoamerican Math Olympiad.

The geometrical situation is simple : Consider two circles k_1 and k_2 , intesecting in two points, and let t be one of their common tangent lines.

The involved points which repeat in the problems are one of the intersecting points, P say; and the points A and B in which the circles touch the common tangent line, more near to P.



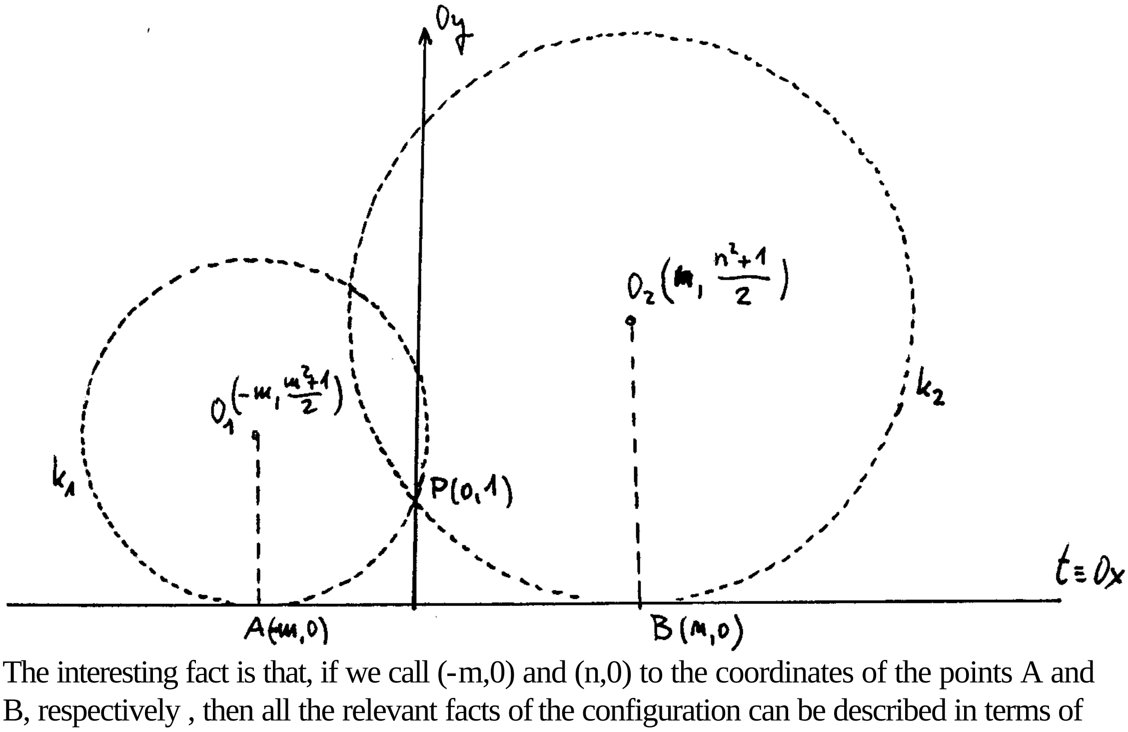
The statement of the problems were the following :

Problem 2, APMO 1999 (march) Circles k_1 and k_2 intersect in points P and Q. The common tangent line, more near to P, touch k_1 at A and touch k_2 at B. The tangent to k_1 at P meet k_2 in C (distinct of P), and the line AP meet BC at R. Show that the circle circumscribed to PQR is tangent to BP and also to BR.

Problem 1, IMO 2000, Taejon, South Korea, july Circles k_1 and k_2 intersect in points P and Q. The common tangent line, more near to P, touch k_1 at A and touch k_2 at B. The line through P parallel to t intersect again k_1 at C and k_2 at D. Lines CA and DB meet at E. Lines AQ and CD meet in M ; lines BQ and CD meet in N. Show that EM = EN.

Problem 2, Iberoamerican Math. Olymp. 2000, Caracas (Venezuela), September
Circles k_1 and k_2 intersect in points P and Q. The common tangent line, more near to P, touch k_1 at A and touch k_2 at B. Let D be the point of intersection of the line joining the centers of the circles with the perpendicular from B to AP. Let C be the diametrally opposed point to B. Show that points P,D and C are collinear.

The synthetical solutions of the three problems can be read in several booklets from the competitions. In the first case one uses the fact that A,B, R and Q are concyclic points ; in the second, calling R the point in which PQ meet t, the similarity between the couples of triangles QAR and MPQ, and QBR and PQN ; and in the third one, several considerations about angles which allow to prove that the angle DPB is a right angle.
The level of difficulty is not very high (usually the really hard problems in a competition are the last), but none of them are easy problems.
And, moreover, another common feature of the three problems : if one try of made an attack by analytical geometry, the reference system formed by the line of centers and the radical axis PQ is really useless, because in this system the equation of the tangent t is not easily expressed by means of the parameters of the configuration. So, until a very recent date, I think really of these problems as an example in which the geometry classical wins to the analytical. I must say that I am the author of a paper titled *The Power of the Analytical Geometry*, published by the Mathematical Society of Catalonia, in which several examples of Olympic problems are solved by the Descartes heritage method, and therefore my feelings after several unsuccessful attempts of attack the third problem were frustrating. But in the issue number 9 of the Brazilian Olympic journal *Eureka!,* (December 2000) the solution given during the contest of the Iberoamerican Mathematical Olympiad by the student Daniel Massaki Yamamoto (Brazilian, yes!) was published, and this solution is in fact analytical.
The key of this solution to the problem form the Iberoamerican Mathematical Olympiad is, of course the clever election of the reference system. Massaki Yamamoto choose the common tangent t as axis of abscisses, and the perpendicular to t passing through P as ordinate axis. Moreover, take the unity of length in such way that the coordinates of P are (0,1) :



The interesting fact is that, if we call (-m,0) and (n,0) to the coordinates of the points A and B, respectively , then all the relevant facts of the configuration can be described in terms of the parameters m and n, and in particular the radiuses of k_1 and k_2 which are respectively equal to $(m^2+1)/2$ and $(n^2 + 1)/2$. For example, the equations of k_1 and k_2 are

$$x^2 + y^2 +2mx - (m^2 +1)y + m^2 =0$$

$$x^2 +y^2 -2nx - (n^2 +1)y + n^2 = 0$$

the radical axis PQ is $2x + (n - m)y + m - n = 0$.

So, the problem from the Iberoamerican Mathematical Olympiad ask for the collinearity of the points

$P(0, 1)$, $D((mn - 1)/(m + n) , m(n^2 + 1)/(m+n))$ and $C(n, n^2 + 1)$

And this is easily checked; in the problem from the I.M.O., the coordinates of the point E are $(0 , -1)$ and it is necessary to check that the two points

$M (2(1-mn)/(m+n) , 1)$ and $N (2(mn - 1)/(m+n) , 1)$

Are at the same distance from E, but this is now trivial .

The computations for the first (chronologically speaking) problem are more complicated, but a program as MAPLE is of valuable help.
The coordinates of the point R quoted in the problem are

$R(2m(mn - 1)/(m^2 + 1) ; (m^2 + n^2 + 2)/(m^2 + 1))$
And those of the circumcenter K of the circle PQR,
$K((mn - 1)/(m+n) ; m(n^2 + 1)/(m^2 + 1))$

And now again is easy to chek that BP is perpendicular to PK and BR is perpendicular to RK.

Finally, a few words for explain why I write the paper *The Power of the Analytical Geometry* as an Olympiad training session : in the educational system of my country, the Geometry has almost disappeared. So, our students only have as geometrical weapons for attack a geometrical problem those proofs of the analytical geometry, which, like Descartes said, allow us solve all the geometrical problems.