Problem Fields in Geometry Teaching

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Working on Problems and discovery learning is demanded since more then fifty years. But in most countries its realisation in school leaves a great deal to be desired. Therefore it is *important to support a teaching which contains a lot of acting with problems*. Also the practice of geometry teaching is not extensive enough though in geometry teaching general goals can be reached very good and the belief about mathematics can be improved.

According to these two points of view I want to present some problem fields for geometry teaching. My focus however is not on separate problems – which you can find e.g. in mathematical Olympiads – but on ***problem fields*** *where the student beside solving problems can make investigations and problem posings as well as find mathematical connections and systems*. I favour this way because it's much more nearer to the complex work of professional mathematicians, gives more opportunities for different general objectives and also can lead to mathematical concepts as well as systematic school mathematics.

Now I want to present some ideas for problem fields in elementary geometry for secondary school which may be a stimulus for your on praxis. The first example is for elementary or junior secondary school to enrich the imagination about shapes of triangles on hand and to train systematic and combinatory thinking on the other hand. The second example has to do with quadrilaterals, its general definition and the ordering of different types. The third example deals with a systematic description of all regular polygons. In junior secondary school it is also interesting to look out for polygons in space which is the theme in the forth problem field. The fifth and sixth examples deal with trigonometry. First we deal with an introduction to trigonometry where the students can start with the problem of computing all sides of a given triangle and then can find connections between different trigonometric functions by themselves as well as connections between regular polygons and special values of trigonometric functions. Secondly in senior secondary school some families of trigonometric functions give us a field of interesting investigations. In the last example then I will give some hints for an applied geometry teaching with problem fields of everyday life.

1. **Different shapes of triangles especially triangles with integers as length**. Mostly in grade five or six the children have to treat triangles and learn different types of triangles like isoscales, right-angled, equilateral and isosceles-right-angled triangles but also acute angled and obtuse triangles. In this context the pupils also should make experiences with very acute angled and very obtuse triangles to get an idea of the big range of possible shapes of triangles.
   A nice point of start for this problem field is the analysis of triangles with integers as length. Such triangles are natural for the pupils because at this time they mostly know only natural numbers. *A first task then is to find all triangles whose sides have a length of 1 or 2 or 3 or 4 inches*. With this task you have a typical mathematical question, the question to find all objects with a special condition. This is also a typical problem of combinatory. You also will find a theorem by working on the problem, namely the theorem that the sum of two lengths is always bigger than the third length, a theorem which can be connected with the definition of a line as the way with shortest distance. Moreover you find that there are triangles with the same shape, they supply only one shape if we do not look on scaling. So the pupils already in this age can build an idea of similarity.
   After solving this task the pupils for example can look out for isosceles triangles with length of the basis equal to 1 and the length of the other sides equal to 1, 2, 3, 4, 5, ... , 10 [or an other natural number n] and reverse the length of the basis equal to 1, 2, 3, 4, 5, ... ,10 [or n] whereas the other sides have a length equal to 1, 2, 2, 3, 3, ... , 6 [(n+1)/2 or (n+2)/2].
   They also can look out for the number of all triangles whose sides measured in inches are an element of {1} or {1; 2} or {1; 2; 3} or {1; 2; 3; 4} or {1; 2; 3; 4; 5} or {3; 4; 5}. Let us discuss for example the problem with {1; 2; 3}, that is we look out for triples (a; b; c) where a, b, c are the lengths of the three sides of a triangle and $a,b,c \in \{1; 2; 3\}$. To reduce the possibilities because of congruence we can set $a \leq b \leq c$. With this and systematic thinking we find that all possible triples are (1;1;1) , (1;1;2) , (1;1;3) , (1;2;2) , (1;2;3) , (1;3;3) , (2;2;2) , (2;2;3) , (2;3;3) , (3;3;3) . But not all of these triples stand for a triangle because of the inequality about the lengths of the sides which comes out here as $a+b>c$. Therefore (1;1;2) , (1;1;3) and (1;2;3) have to be

kicked out. Also we can see that (1;1;1) , (2;2;2) and (3;3;3) lead us to equilateral triangles. Therefore not looking for the size these three triples give us only one type of triangles. Beside this type we have three acute isoscales triangles – (1;2;2) , (1;3;3) , (2;3;3) – and one obtuse isoscales triangles – (2;2;3) .

If you work with pupils of higher grade or students of a college respectively university a nice task is also the analysis of all triangles whose ratios of the measure of angle are 1 or 2 or 3 or 4. For example the ratios 1:1:1 lead us to the equilateral triangle and the ratios 1:1:2 or 1:1:3 or 1:2:3 for angles are possible and lead us to triangles with 45°, 45°, 90° (half of a square) or 36°, 36°, 108° (part of a regular pentagon) or 30°, 60°, 90° (half of an equilateral triangle).

2. **Definition of quadrilaterals and systematising plane quadrilaterals**. Normally in grade six or seven the pupils are confronted with quadrilaterals. Mostly the instruction begins with special types like squares, rectangles and parallelograms. But after that the question should come out: How can we define a quadrilateral in general? Normally we will say that a quadrilateral is characterized by four different points which are connected by four lines so that these build a closed line-sequence. But then we can discuss whether we call such a figure with three of the edges on a line also a quadrilateral or not. After that discussion *we can distinguish between quadrilaterals in the plane which have crossing sides or a concave shape or what is most known a convex shape*. Further we can look out for symmetric and oblique-symmetric quadrilaterals. If we do that for the convex quadrilaterals we will get an *ordering tree with the square at the bottom, followed from rectangle, rhombus and symmetric trapezium, parallelogram, kite in the next two rows. Then we have the general trapezium*. How can we bring it in our tree? There are different ways possible. If we look on equal sides on hand and right angles or parallel sides on the other hand a quadrilateral with two equal sides would fit together with the trapezium in one row. But if we look on symmetry the row better should be characterized by skew-symmetry because the trapezium has a oblique-reflection-line through the midpoints of the two parallel sides. Then we find as partner for the trapezium the so-called oblique-kite which has a oblique-reflection-line through two opposite edges. It also can be characterized as quadrilateral where one diagonal cut the other one in two halves.
3. **Systematization of regular polygons in the plane**. Regular polygons in the plane we can get for example with a computer procedure that allows to repeat the task "go from the point you are now standing in a given direction a given distance and then change your direction with a given angle δ ". Starting with this procedure we can find that with a special angle for changing direction we get a regular polygon. If $\delta = 360°/n$ then after using the procedure n times (and a total turning on 360°) we come back to our starting point so that our path describes a convex regular polygon with n edges. But it is also possible that we come back to our starting point after more than one 360° turnings. In these cases we get regular polygons with crossing sides. After having made such experiences by more analysis we can find out *that all regular polygons of the plane which can be drawn without interruption* emerge from our procedure with $\delta = 360° \cdot k/n$ *, where k, n are natural numbers with* $0 < k < n/2$.

   We also get regular polygons for $k < 0$ or $k > n/2$, but only the ones we already got before, however run through in a different way. Moreover: If $k_1/n_1 = k_2/n_2$ then we of course get the same angle δ. But it is also possible to make a one-to-one-relation not only to the ratio k/n but also to *all pairs (k ; n)* if we include regular polygons which can not be drawn without interruption. For example the pair (1;3) leads us to $\delta = 120°$ and therefore stands for the regular triangle. Then (2;6) should stay for a regular polygon consisting of two regular triangles, namely the so-called David-star used in Jewish culture. And (3;9) will stand for a regular 9-edged polygon consisting of three regular triangles with the same midpoint and twisted to each other with 120°. and so on.

   We can enlarge this theme by looking out for half-regular polygons. That means for polygons where the angles are all equal but for the length of the sides there are two numbers changing. Or we have all sides with the same length but the measure of the angles change constantly between two values.
4. **Polygons in plane and in space**. Normally in school but also in university in elementary geometry we think of polygons in the plane. But if we define a polygon as a closed line-sequence

then it is also possible to look for polygons in the space. I would emphasize on such a problem field because on one hand we can train our imagination of space with that (which is very important and done too seldom) and on the other hand we can extend the knowledge about three-dimensional shapes whereat a polygon in space means that it does not lie in a plane. Therefore first we will look at *paths through edges of different solids which built a polygon in space*. With this an interesting point is the look out for different paths which are congruent. For example through the edges of a regular tetrahedron we find only one type of polygon in space. It is a regular quadrilateral where the mearsure of an angle between two sides which have a point in common equals 60°. On a square pyramid we can find two types of polygons in space, a quadrilateral and a pentagon. We can enlarge these investigations by using midpoints too.
Then we can ask for *regular polygons in space*. The definition can be found by transfer from the plane. That is: All sides are congruent and all angles built from sides which have an edge in common are congruent. First it's clear that there exists no triangle in space. A regular quadrilateral in space we can find on the tetrahedron as already seen. But the angle of a regular quadrilateral in space must not be equal to 60°. If we change all four angles of the regular quadrilateral found on the tetrahedron in the same way we also get a regular quadrilateral in space. Moreover we can show that all possible regular quadrilaterals in space can be built in this way where the mearsure of the angle lies between 0° and 180°. For n = 5 we will find no regular polygons in space and a systematic for n > 5 is very difficult because of many degrees of freedom. For n = 6 for example we can find on a cube a regular polygon as well as on the curves surface of an anti-prism, where each time three diagonals build a regular triangle on top and on the bottom of the anti-prism. Moreover for each even n we find a regular polygon in space on any anti-prism with a regular plane polygon of n/2 edges on top and on bottom.

5. **Genetic orientated introduction of trigonometry**. Mostly trigonometry begins with the functions sinus and cosinus introduced on a right-angled triangle or on a standard circle. Then this has to be given by the teacher. But an introduction of trigonometry can also come out of a problem. From the theorem of Pythagoras the pupils have seen that with given right angle and the measure of two sides we can calculate the length of the third side. From earlier time we know that the triangle then is fixed without looking for congruent figures. Also we know (and already Thales knew) that *a triangle is fixed in this way if two angles and one length of a side is given*. We already have found out the other measurements of such a triangle by constructing a similar figure and measuring it. But with this method we only get approximations. Therefore the question can come out if there is a possiblity to get them exact by computing in analogy to the application of the Theorem of Pythagoras. By looking out for a solution of this problem *a view into history can help us* (Thales and Pythagoras are hints to the history).
Through an investigation of history of mathematics or with a hint by the teacher we find ideas about ratios of sides with the Egyptians and their pyramids more than 4000 years ago, with Aristarch and his calculation of the distance of the moon about 150 B. C. and especially *Ptolemaios (about 150 A. C.) who can be seen as the founder of trigonometry*. He already worked out a table of values for a trigonometric function. But he had only one function because he used isosceles triangles. The Indians (about 400 A. C.) then went over to the right-angled triangle, the half of the isosceles triangle. *During this historic excursion we were lead to the statement that all isosceles triangles with equal angles in the top are similar* so that the ratio of their sides is beside 1 only one fixed real number. Also all right angled triangles with equal angles are similar. After discussing these statements it is a good method *to define all six trigonometric functions (sinus, cosinus, tangens, cotangens, secant, cosecant) namely as all six possible ratios of the three sides simultaneously*. Then the pupils immediately can come to connections between these six function by themselves.
The next question after that will be *the question about the values of such a function*, for example sinus (for the others we know already the possibility of conversion). Here we now have to remember that with some regular polygons we know all angles and all ratios of sides and diagonals. With looking at a half regular triangle and a half square we find the values of sinus for 30°, 60° and 45°. We also could find values for 18°, 36°, 54° and 72° at the regular pentagon. But then we have to learn from Ptolemaios that we can find a formula for sinus(α/2) and sinus($\alpha+\beta$) which give us possibilities to get the value of sinus for a whole list of angles. This was the way

mathematicians from Ptolemaios until the middle of the last century got the values of the trigonometric functions. Today inside of the computers this values are determined by the Taylor-function of the trigonometric function. But this we only can tell the pupils at that time and create interest for further mathematics.

6. **Families of trigonometric curves**. A family of curves can be defined as a set of curves which have most qualities in common and one or some quality varying. With coordinates they can be described by curve-equations which have one or more parameters. For example the set of all straight lines with one direction (e. g. with equation $y = x + b$ and b varying) or all normal parabolas through one point (e. g. with equation $y = a \cdot x^2$ and a varying) builds a family of curves. *The aim of working on families of curves on one hand gives a new view on a special type of curves (for example a dynamic view or the embeded curve) and on the other hand through systematic varying gives possibilities to find connections and special attributes.*
   The above named families are not very proper for theses aims. Therefore I propose here the *family of trigonometric curves. It has also relevance in function theory, electro-dynamics, oscillation theory and analysis of music*. Some problems we can work on are the following:
   i) We know that with the transformation $x \to x + \pi/2$ the sinus-curve goes over into the cosinus-curve. What do we get by the transformation $x \to c \cdot x$ with $c = 1, 2, 3$ or π for example?
   ii) From the physical analysis of tones we know the superposition of sinus curves. What can we find out (with help of a computer) about the curves out of the family with equation $y = a \cdot \sin(b \cdot x) + c \cdot \sin(d \cdot x)$ and $a = b = 1$, $c = 0.1, 0.2, 0.3, 0.4, 0.5$, $d = 5, 4, 3, 2, 1$ for example.
   iii) The three curves with the equation $y = \cos x$, $y = \cos x + 3 \cdot \sin x$, $y = \cos x - 3 \cdot \sin x$ for $0 \leq x \leq \pi$ build a nice figure like a modern pair of glasses. Find a symmetry or an other transformation of this figure into itself! Can determine the circumference or the area? Can you generalize these results?
   iv) If you use the sinus function not in cartesian coordinates but in polar coordinates you will find also very interesting curves. For example how do the curves with equation $r = |a \cdot \sin(b \cdot \varphi)|$ with $(r;\varphi)$ as polar coordinates look like? What properties can you find for special a and b ? With which a and b are these curves not infinite?
7. **Geometry in everyday life**. With help of geometry we also can solve several problems of everyday life. For this I emphasize on problem fields which are given through a situation of everyday life.
   For example a family plans to build out the loft. Then there are several problems to solve. Or a piece of a dyke has to be renewed, a situation for people working in the community. Here also several problems including plane and three-dimensional geometry have to be solved. An other problem field within this frame is the conscious discussion of symbols, shapes and architecture in the middle ages. But because the theme “Geometry in everyday life” is to big for this paper I will remain with these hints. For more details see my paper in Research Report 55, Department of Teacher Education, University of Helsinki, 1987 or in Blum et al. (Ed), Applications and Modelling in Learning and Teaching Mathematics, Chichester 1989.

I hope the short introduction and the named seven problem fields gave a picture of my idea about problem orientation in geometry teaching and leads to a vivid discussion.