Algebra in the head, on paper and on the screen

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Abstract: This paper briefly reviews the nature of algebra and considers how the development of understanding depends on the context of application and medium of representation. It outlines a framework for planning and analysing learning activity and examines the results of research into the use of spreadsheets and other modelling tools in the teaching and learning of elementary algebra. It draws conclusions concerning the influence on pupils' abilities in algebra of teaching methods and tasks set, and concerning the role of ICT in developing 'abstract' thinking and suggests that:

- reflective activity must be incorporated into learning tasks in order to develop algebraic thinking;
- symbolic representation using computer modelling tools provides experience to be reflected upon, rather than being algebraic in itself;
- the success of the teaching is influenced in particular ways by the tasks set, the tools provided and the teacher's orchestration of the features of the setting for learning.

Finally, issues are raised concerning the value of learning standard algebraic syntax in a world where mathematical modelling is predominantly carried out using computer systems.

Algebra has always been considered difficult to learn, and correspondingly hard to teach. There are complex issues involved in considering why this should be the case. Teachers intuitively feel that the problems seem to result from the extra level of abstraction involved in algebra compared with purely numerical work. It is important to help pupils overcome these difficulties, however, for "… there is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will soon render it impossible to make hard things simple." (Tall & Thomas, 1991: 128).

A considerable body of evidence has been assembled which attests to the difficulty of learning algebra, and rather than seeking ways of overcoming the difficulties, schools have delayed and restricted the algebra curriculum (Sutherland, 1997). There are other ways of tackling this matter, however. One is to exploit the clear potential for digital technology to bring about fundamental changes in how mathematics is developed and used. It is less obvious in what ways ICT will affect how mathematics is taught and learned, but it seems essential to attempt to employ this powerful medium in the pursuit of extending the algebra curriculum.

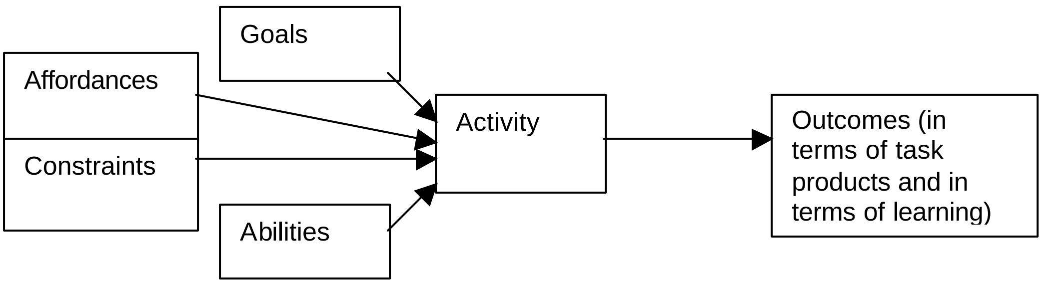
A number of features or functions of ICT have been suggested by the Teacher Training Agency (TTA, 1998) as the most important in improving learning: capacity and range; speed and accuracy; provisionality; interactivity; and it is the last of these which seems to offer most in the teaching of mathematics. It has enabled the development mathematical representation systems which 'share properties with the world of physical objects and hence can tap into that wealth of processing power and competence that develops from normal human development apart from school' (Kaput, 1994: 388). This has profound implications for the learning of mathematics. However, we must also address the concern that 'the inherent structures of most microcomputer approaches to representation may constrain, distort, shortcut or undermine these productive processes' (Kaput, 1994: 395).

We shall consider these issues in more detail, examine some attempts to exploit computer modelling tools in the learning of algebra, and discuss some implications for mathematics and its pedgogy.

Nature and development of algebra

One view of algebra is as the manipulation of symbols according to fixed rules. With this perspective, it is natural to think of the symbols as objects; they have a visible character which renders them concrete. Algebra should be quite easy to learn because of its predictable nature. Such a view is naïve, however: the symbols are, after all, mere signs. The actual objects signified are variables (or functions), which are mental objects having no concrete form. We cannot experience them directly through any of our senses. We have to construct them in our minds, and this is not a trivial process. Although we have devised paper representations for mathematical phenomena, these representations are not the entities themselves, and can represent only the state of the entity at a particular instant, with no change and no interaction. Furthermore, a novice who is trying out ideas cannot gain feedback from interaction with the paper representation in the way that a learner of science can gain feedback from the environment, and the mathematics learner must usually wait for a teacher to

provide a response to their efforts. Although group activity can ameliorate this situation by facilitating social construction of mathematics amongst peers, this can result merely in sharing misconceptions, and it is natural that we should seek other means of providing learners with immediate expert feedback on their ideas.
Much of the development of mathematics has been analysed by Sfard (1995) in terms of *reification*: the transformation of a mathematical process into an object of thought which can in turn be operated upon in more advanced processes. This idea seems particularly appropriate in characterising the transition from arithmetic to algebraic thinking. Algebraic thinking involves handling the unknown mentally, and working from the unknown to the known in solving problems. Arithmetic thinking, conversely, involves working from the known to the unknown. Thus an equation such as 3x+2=x-1 may be solved arithmetically by trial and improvement, or algebraically by operating on the expressions involved according to rules known to be valid for these mathematical objects. Sutherland (1994: 276) makes a general observation that 'the emphasis on structure in algebraic thinking can be contrasted with an emphasis on process in arithmetic thinking. Algebraic thinking does not replace arithmetic thinking – it supersedes it, becoming a new vantage point from which to view arithmetic.'
We can also take the view that the transition from arithmetic to algebra 'is not initiation into decontextualised knowledge but initiation into another social practice' Sutherland (1991: 45). In this view, generalisation and abstraction are not processes or states of mind, but changes in practice which occur within a relevant setting. Learning algebra thus involves immersion in mathematical culture, and 'getting used to' algebraic language (Freudenthal, 1973). The transition from arithmetic practices to algebra practices represents a change in goals, tools, and ways of working. Although the participants are the same, it is a new setting with different cultural values – almost like moving from subject to subject or school to school. Learners hold apparently contradictory views, because they see them as valid explanations for different circumstances (see for example, Herscovics, 1984). In this view, the teacher who, apparently pointlessly, writes 'algebra' on the board at the start of the lesson is in fact defining the nature of the setting for pupils. This act will have a very special significance if the algebra lesson takes place in the ICT room, which possesses its own culture (Kennewell et al., 2000).
It is also important to adopt a 'situative' approach (Greeno and Moore, 1993), in which the unit of analysis comprises a setting for action and a set of goals as well as the individual agent. The *affordances* of a setting in a situation constitute the potential for action inherent in the features of the setting. They are the properties of the things and materials in the situation that create support for particular activities. Affordances can take different forms: static features of the setting, such as a poster displaying geometric figures; dynamic features, such as videotape of shapes growing with sides in constant ratio; tools, such as rulers and pencils; humans, such as teachers, classroom assistants, peers.
We also need to note that the setting imposes *constraints* – relationships and conditionality in the setting which determine the nature of action, limit its scope of action and influence its course. Constraints should not be viewed in a negative sense of inhibiting action; in many cases, the structure imposed by the setting may facilitate the course of action. Indeed, it is common for educational settings to be deliberately constrained to facilitate action on the part of novices.



The potential for action in a setting does not guarantee that the agent will carry out the action successfully. Greeno and Moore (1993) point out that the role of affordances in an activity is relative to the *abilities* of the agent – the knowledge and disposition that enables them to engage in activities. The affordances may not be adequate for goal-directed action to be taken, so that the agent is 'stuck'; or they may afford inappropriate action, so that the agent makes errors. Action is successful in

attaining goals if the combination of abilities and affordances are adequate in combination. They work jointly during activity to determine the course of action.
Learning is then characterised as an increase in abilities. This will take place under two main conditions:

- the initial affordances are inadequate to enable learners to reach their goals using their current abilities, so that further temporary affordances are sought by the learners or provided by a more able person;
- learners adapt other abilities through trial and error, and refine these so as to become appropriate for the tasks they engage in.

The learning will be enhanced if the teacher stimulates reflective activity in which learners make explicit the affordances and constraints and consider the scope of their invariance – in other words, they seek to generalise the features and relationships of the setting. This reflective stage supports concept development which enables better utilisation of the learning in new situations.
The construct of affordance may be used to characterise the design and implementation of teaching situations in which teachers typically plan, implicitly or explicitly, to orchestrate the affordances and constraints (either directly or though learning materials in a teaching situation so that:

- the affordances are initially insufficient to enable the learners to be successful in their activity on the basis of their current abilities;
- the affordances can then be increased by the teacher to enable the learners to succeed - an information sheet, a calculator, simpler language, a question, a demonstration, an analogy, a brainstorm, a discussion group, pairing with a more able student;
- learners' abilities will change as a result of this experience.

Thus, when a teaching situation involves computer use, we need to consider:

- what the computer affords the learner in carrying out activities;
- how the computer's affordances relate to other affordances of the didactic situation;
- how the affordances may be manipulated by the teacher so as to promote learning.

We also need to ensure that:

  - learners have adequate abilities in using ICT
  - metacognitive activity is encouraged during the task
  - reflection on the activity is stimulated

(Kennewell et al., 2000; Kennewell, 2001)
The visual displays of computer systems potentially afford activity that mere paper does not. However Mason (1995) suggests that, whereas words are generalising media, screens are particularising media, and something extra is needed to indicate the idea of generality in the case of a screen image. With the use of a mouse, however, there is the possibility of kinaesthetic image schemata: based on a bodily sense of variation and visual detection of what is involved during change on the screen. Mason (1995: 123) characterises this phenomenon as 'mouse mathematics'. Sutherland (1990) suggests that a formal rather than natural language is required for expressing generality, and that the computer-based tools provide this formal language which affords the expression of generalisation in a way that natural language does not.
The conditions highlighted by Mason and Sutherland are both found in computer modelling systems. The main tool for computer modelling taught in schools in the spreadsheet program, and it seems sensible to consider this software as the prime candidate for curriculum activities (Kennewell, 1997). It has limitations deriving from its design as a business tool, however, and its intrinsic character is not fully algebraic (Dettori et al., 1995). For instance, it represents a literal variable in a formula as a generalised number at best, and the variable is more likely to be seen as a specific value. There is a danger that pupils are actually learning to avoid algebraic thinking by using the spreadsheet, but the spreadsheet environment does provide sentential as well as visual systems and 'pupils point to the screen to communicate mathematical relationships to their partners and encapsulate this visual relationship by pointing and clicking with the mouse. They also simultaneously use the spreadsheet language to communicate their ideas…' (Sutherland, 1992: 78).
Sutherland (1993) further noted that the spreadsheet is enabling pupils to negotiate a symbolic representation which has meaning for them, rather than merely trying to translate a problem syntactically from words to algebra. She found that pupils who had carried out a unit of work using

only spreadsheets were able to utilise this knowledge in a paper based post-test, by using spreadsheet notation to represent variables and functions. She suggests that 'this symbolic language has taken on an important mediating role' (Sutherland (1993: 45). These pupils had been stimulated to think about mathematics in a way that was unfamiliar to them:

'RS: … Why is it that you can do it now when you are on paper?

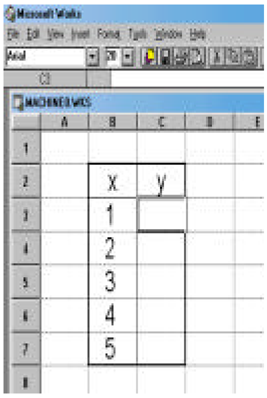
Jo: Because you have to think before you type into the computer anyway'.

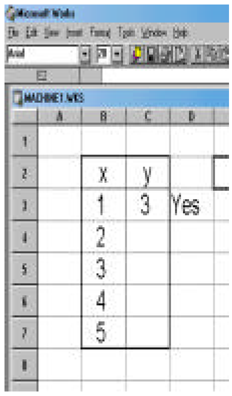
Jo subsequently said she knew she was thinking because her 'brain hurt'.

Pupils tend to see the process of entering a formula and replicating it down a column as two parts of the same operation: 'Thus they have an image of a physical location, not only for the physical cell into which they enter a particular number, but also for the column of cells into which they may enter a whole range of numbers' (Ainley, 1995: 32). The use of columns (rather than rows) to represent variables has become quite standard in the literature on spreadsheets in mathematics education. There is some justification for this consistency, in that it provides a constraint additional to the basic spreadsheet features to which is easier for pupils to become attuned.

Investigating the computer modelling in teaching algebra

The potential of computer modelling systems in the learning of algebra was investigated in a phased study, carried out over a number of years in various real school contexts as the availability and teaching of spreadsheets increased during the 1990s (Kennewell, 2001). A variety of strategies for employing these tools pedagogically was explored. Central to the later phases of this investigation was the idea of a 'number machine', for example; this is a representation of the idea of mathematical function commonly used by teachers. It is possible to exploit the interactive potential of the spreadsheet and turn a whole class number machine activity into a personal challenge. Pupils can be presented with the table (to be known as a 'machine') as shown on the left, and asked to guess (or work out when they suspect a rule) a value for *y*. If successful, *Yes* will appear (as shown on the right), otherwise *No*. To avoid pointless random activity, they may be told the first value to use for each function.





The studies confirmed that objects-to-think-with can be provided by ICT and created by pupils using ICT.

Construction of formulas is afforded by visual features on the screen, mouse and keyboard - cells, symbols, point-and-click - which support the step-by-step thinking characteristic of the operational view of function. Symbolisation has a clear purpose in a computer modelling environment and becomes a natural process, but carefully designed tasks are required to shift pupils' thinking to structural features of algebraic expressions. Furthermore, the syntactical differences between computer modelling systems and conventional algebra mean that the influence of computer modelling activity on the ability to learn conventional algebraic representation is not wholly positive, however. Spreadsheet syntax strongly influences pupils' initial attempts with algebraic syntax, and pupils may benefit from earlier introduction of algebraic notation and specific requirements concerning the syntax to use in computer representations. Spreadsheets may not be the most effective computer modelling tools for the purpose of teaching and learning algebra; some dynamic modelling tools are easy to learn, and may offer advantages in relation to learning algebra.

We found that pupils expected to get results immediately with ICT and needed prompting to use mathematical strategies other than trial and improvement. One effective approach was to teach strategies for identifying linear functions from tables of values prior to spreadsheet work. Structured exploration of computer models can help develop the concepts of variable and function, but a variety of levels of understanding remain, and some aspects of leaenrs' abilities are situated within the computer modelling environment and not applied to algebraic contexts. The understanding of structural relationships between algebraic expressions, which form the basis of algebraic manipulation, does not develop purely though computer modelling activities which privilege the generation of numeric results using operational thinking. Reflection on the structural relationships intrinsic to the real number system is required, and the place of computer modelling activities in the development of algebraic abilities should be to provide experiences upon which to reflect and the disposition to do so.

There were clear indications that the process of reflection is a vital element in bringing about the abstraction required for developing algebraic abilities, and that ICT in itself is insufficient. Computer modelling systems afford too many actions and do not constrain sufficiently to ensure that learners encounter key experiences on which they will reflect. The provision of constraints which will stimulate reflection, then, is a key didactical role of the teacher in an ICT environment. Pupils must be placed in situations where their intuitive methods prove unviable, so that they can be challenged to review their ideas about the formal system. This has implications for the design of tasks. The more successful pedagogical strategies required sustained metacognitive activity by pupils, with autonomous control of task and learning progress together with whole-class interventions to focus pupils' thinking on relevant features of their activity. It is difficult to balance the pre-planned pace of lessons based on whole-class teaching with the opportunities for pupil decision-making afforded by ICT, however. Some change in pedagogy is required from both the current ICT teaching environment based on 'computer culture', and from the whole class teaching typical of traditional mathematics lessons.

Implications for algebra and algebra education

Algebraic activity has developed over many centuries through the interaction of mind and representations on paper. This process has produced a syntax which is well-suited to paper-based activity, having a principle of economy with notation which affords manipulation of expressions by those with appropriate abilities. The screen and pointer have different features from paper and pen; the facility to point and click or drag on a displayed symbol or expression rather than constructing it mentally and physically has significant implications. There seems to be a new form of process-object duality: instead of the idea of function developing as an object from a process of calculation, it may now derive more directly from the process of symbolic formula construction and replication. Indeed, a new system of notation could be designed, based on the principle of 'learnability' rather than economy of expression. Such a development is not likely to succeed in the field of mathematics, however, any more than a spreadsheet which affords algebraic activity is likely to succeed in the business environment. Each existing system is highly entrenched in its own field.

What we should thus attempt to do is to make the best use in mathematics teaching of tools which pupils are expected to learn for other purposes. To this end, the following sort of teaching programme is recommended.

1. Initial exploratory modelling activity which:
- introduces the essential features of the computer modelling system;
- involves structural thinking about arithmetic operations and their inverses, and the relation between multiplication and addition;
- requires hypothesis generation and testing.

2. A series of lessons, each with a specific set of objectives which all pupils are expected to achieve and a set of tasks which pupils will undertake though the use of ICT. The objectives should include:
- the representation of linear functional relationships symbolically using words, number machines, and conventional algebra;
- the representation of such relationships using spreadsheet formulas;
- challenging misconceptions concerning the conventional order of operations, the associativity/distributivity of multiplication and addition, the role of brackets;
- changing the form of expressions involving multiplication and addition to produce equivalent expressions with the operations in different orders.

3. Opportunity for open modelling tasks which develop pupils' abilities in representing relationships and allow them to develop their metacognitive functioning with support from teacher questioning and whole-class discussion of strategy.

The design of support resources should also aim to take advantage of pupils' growing abilities in navigating and interpreting complex information on the screen, which may be the best place for task instructions, questions, and feedback from the teacher.

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